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Peeping into Fibonacci's Study Room

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Abstract

The following collects observations I made during the reading of Fibonacci's *Liber abbaci* in connection with a larger project, "abbacus mathematics analyzed and situated historically between Fibonacci and Stifel". It shows how attention to the details allow us to learn much about Fibonacci's way of working. In many respects it depends critically upon the critical edition of the *Liber abbaci* prepared by Enrico Giusti and upon his separate edition of an earlier version of its chapter 12 – not least on the critical apparatus of both. This, and more than three decades of esteem and friendship, explain the dedication.

Contents

Per Enrico Giusti

It is conventional wisdom that Leonardo “of the house of the sons of Bonacci”, outside Pisa identified as Leonardo Pisano and since the 18th century for short as Leonardo Fibonacci, learned to use the Hindu-Arabic numerals in Bejaïa in present-day Algebra and brought them to Christian Europe; partially false, of course, since al-Khwārizmī’s treatise on the topic had been translated into Latin early in the 12th century and a number of redactions produced in subsequent decades[1] (not to speaks of the ciphers written on the counters of the “Gerbert abacus” since the later 10th century). Fibonacci is also often supposed to have been the one who brought Arabic mathematical knowledge to Christian Europe – again partially false, since the 12th-century translators had done much even on this account.

The basis for these partially sound, partially false assumptions is this part of the prologue to the *Liber abbaci*:[2]

> When my father, appointed by his homeland, held the post of public *scriba* (notary or representative) in the custom-house of Bejaïa for the Pisan merchants frequenting it, he arranged for me to come to him when I was a boy and, because he thought it would be useful and appropriate for me, he wanted me to spend a few days there in the *abbaco* school,[3] and to be

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[1] Since this is well documented and not my topic, references to [Allard 1992] and [Folkerts 1997] should suffice. Less familiar is the use of Hindu-Arabic numerals by a notary in Perugia from 1184 onward – they stand in the beginning of documents and indicate the number of lines contained in the document [Bartolo Langeli 2000 :230–244]. From where this notary learned them is unclear, but in any case he did.

[2] I give Charles Burnett’s translation [2003: 87]. Further translations, if no translator is identified, are mine.

[3] “*Abbaco* school” renders *studio abbaci*, which may as well mean “study of the abbacus” – and the ensuing “art”, referring to the *abbaco,* supports the latter reading. It cannot be concluded from Fibonacci’s words that he frequented something like the later Italian “abbacus school”. We have indeed no information about how this kind of practical arithmetic was taught in the Arabic world – be it then that Ibn Sīnā’s father “sent me to a vegetable seller who used Indian calculation” [trans.

> taught there. Here I was introduced to that art (the *abbaco*) by a wonderful kind of teaching that used the nine figures of the Indians. Getting to know the abbaco pleased me far beyond all else and I set my mind to it, to such an extant that I learnt, through much study and the cut and thrust of disputation, whatever study was devoted to it in Egypt, Syria, Greece, Sicily and Provence, together with their different methods, in the course of my subsequent journeys to these places for the sake of trade. But I reckoned all this, as well as the algorism and the arcs of Pythagoras, as a kind of error in comparison to the method of the Indians (*modus Indorum*). Therefore, concentrating more closely on this very method of the Indians, and studying it more attentively, adding a few things from my own mind, and also putting in some subtleties of Euclid's art of geometry, I made an effort to compose, in as intelligible a fashion as I could, this comprehensive book, divided into 15 chapters, demonstrating almost everything that I have included by a firm proof, so that those seeking knowledge of this can be instructed by such a perfect method (in comparison with the others), and so that in future the Latin race may not be found lacking this (knowledge) as they have done up to now.

Whatever the precise meaning of this it is clear that Fibonacci continued to learn on commercial travels to Egypt, Syria, Byzantium, Sicily and Provence. If read to the letter he also seems to have known the (Latin) *algorism* as well as the "Gerbert abacus" (the "arcs of Pythagoras") before going the Bejaïa. At least he knew them when writing the prologue, almost certainly in 1201/02 (see below).

One may ask – [Burnett 2003] raises the question – what distinguishes the algorism from the method of the Indians. The passage from "Therefore, concentrating" onward, contains the probable answer: only the first five chapters, *ca* 10% of the whole work,[4] deals with matters covered by algorism treatises. Chapters 6 and 7 take up calculation with common fractions, whereas algorism treatises (meant to serve astronomy) are only

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Gohlmann 1974: 21]. However, that was some 200 years ago and some 5000 kilometres to the East, and may not have much to say about the situation in the Maghreb in the late 12th century.

Burnett, as can be seen, uses the Italianized *abbaco* for *abbacus*. In what follows, I shall use the Anglicized *abbacus* for the Latin as well as the Italian term – keeping the geminated *bb* to distinguish the concept from the *abacus* referring to some kind of reckoning board.

[4] This number is calculated on the basis of the revised "1228" version, where more may have been added in chapters 6–15 than in the first books. None the less, as will be clear below, 15% should be a firm upper bound.

interested in sexagesimal fractions. The commercial, recreational and algebraic matters contained in the following chapters have no counterpart in the algorisms. What comes from Euclid in these chapters is easily separated out, and the "few things from my own mind" will have been, exactly, *few* (we shall see this confirmed). So, "the method of the Indians" refers to much more than the algorithms for addition, subtraction, multiplication, division, halving, doubling, root extraction and arithmetical progressions – the topics of the algorism genre. It is not certain that *everything* in chapters 6–15 is meant, but the words "concentrating more closely on this very method of the Indians" suggests that the reference should encompass the larger part of what they contained in the first version of the book. Here, the passage "I was introduced to that art by a wonderful kind of teaching that used the nine figures of the Indians. Getting to know the abbaco pleased me far beyond all else" suggests that "art of the Indians" and "abbacus" are more or less synonyms. Then, the introduction to chapter 13, teaching the double false position, turns out to be informative. It claims [B318;G499][5] that by means of this method *fere omnes questiones abbaci solvuntur*, "almost all abbacus questions can be solved". Then, in the section of the chapter that presents alternative solutions to problems that have already been dealt with, everything refers to the collection of mixed, mostly recreational problems from chapter 12. These "flowers" of commercial arithmetic thus appear to belong to the "art of the Indians".

Why then "method of the Indians"? The ambience where the young Fibonacci encountered the notion may have been aware that not only the "nine figures" were of Indian origin. So was, indeed, the Rule of Three,[6] and other methods were also shared within the trading network spanning the Indian Ocean, and for chronological reasons even here there may have been Indian primacy (Arabic trade prior to the *Hijra* was caravan trade

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[5] [B*m*;G*m*] refers to p. *m* in [Boncompagni 1857] and p. *n* in Enrico Giusti's new critical edition. Since the latter is likely to be found at present in only a few libraries while several good scans of the former are available (at least for the time being) at Google Books, these double references seem mandatory. Readers are exhorted, however, to press their libraries to acquire it – also because much of my argument builds on Giusti's critical apparatus. Boncompagni's edition is based on a single manuscript (**F**, Florence, BNC, Conv. Soppr. C.1.2616), probably the best but not perfect.

[6] See [Høyrup 2012] for Arabic and *abbacus* links the Indian vernacular traditions.

within the Arabic Peninsula).[7] On the other hand it is not excluded that Fibonacci himself may be responsible to the generalization. In any case he saw that what *he* presented on the basis of the "Indian" figures went far beyond what was done in algorism and on the Gerbert abacus.

Fibonacci has little more to say about from where he got his material and what he did to it. Euclid is mentioned often in the book, but far from every time we recognize Euclidean material (we shall return to that observation). In one place [B119;G206] Ptolemy and Aḥmad ibn Yūsuf are referred to in connection with the composition of ratios, and the manuscript **V**[8] introduces barter on fol. 47r with the words *hic incipit magister castellanus*, "here begins the Castilian master". Finally, three problems are said ([B188;G319], [B190;G324], [B203;G340]) to have been proposed to Fibonacci by unspecified Byzantine masters[9] while one problem is said [B249;G405] to have been *nobis proposita a peritissimo magistro Musco constantinopolitanus*, "proposed to us by the very skilful Byzantine master Muscus[10]". That seems to be all.

Recently, Nicoletta Rozza has presented a supposed discussion of "the sources of the *Liber abbaci*" [Germano & Rozza 2019: 61–70]. Unfortunately, however, she gives nothing but

> the titles of a large number of mathematical works circulating somewhere in the Arabic or the Latin world during the 10th-12th centuries, but offers no evidence that Fibonacci knew or used them, beyond two references to

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[7] We should perhaps not forget the half-suggestion of Lam Lay Yong ([Lam & Ang 2004: 183*f*] combined with [Lam 1977: 329]) that the Hindu-Arabic numerals were transmitted via different channels, and that the *ghubār* shapes (those apparently used by Fibonacci) were transmitted together with the Rule of Three via the maritime trading network – an idea that does not depend on her argument that the ultimate root of both was in China.

[8] Vatican, Pal. lat. 1343, from the late 13th or the early 14th century [Giusti 2020: xxxi].

[9] The reference to Constantinopolitan "masters" probably means that some kind of teaching institution similar to the later Italian abbacus school was already present in Constantinople. The implications of the reference in the prologue to the *magisterium*, "teaching", received by Fibonacci in Bejaïa are less certain.

[10] Probably "Moschus", but evidently not the ancient bucolic poet, nor any other Moschus I have been able to find.

Menso Folkerts' work concerning Fibonacci's use of Euclid.[11]

If we want to know more about Fibonacci's sources and his way to use them, we need to analyze his text in detail and depth. The text, indeed, gives much indirect information: Fibonacci does not divulge, but also does not mask how he works.

Two versions, their relationship, and a master copy

The *Liber abbaci* is usually assumed to exist in two versions, one from 1202, the other from 1228. With a small and unimportant proviso, the former date seems certain: it is given in all manuscripts that include the prologue. The proviso is that the Pisan year was defined by the Incarnation, meaning that Pisan "1202" lasted from Julian 25 March 1201 to Julian 24 March 1202. The "true" – that is, Julian – date may thus well be 1201. For convenience, however, I shall speak of the first version as from 1202.

As argued by Giusti [2020: xvii], 1228 is more doubtful. Two of the manuscripts of certain or possible 13th-century origin (and one more) state that "here begins the *Liber abbaci* composed by Leonardo the Pisan of the sons of Bonaccio and corrected by the same 28 (or XXVIII)", while the last manuscript of possible 13th-century date has "... corrected by the same in the year 1228", which could be a copyist's interpretation of the shorter variant. No other manuscript gives discordant information, but indicating a year by *XXVIII* or *28* alone would be quite unusual. As we shall see, however, the second version cannot be earlier than 1226. For convenience I shall speak of it as from 1228 – this time with the double proviso that "1228" actually means "no earlier than 1226 and probably some years later", and (as I shall argue) that it may actually have existed in several variants.

With a major and a possible minor exception, all extant manuscripts represent the "1228" version, in which Fibonacci states to have "added some necessary matters and cut out certain things that were superfluous" [B1;G3]. The major exception, discovered by Giusti [2017], is that chapter 12 in the manuscript Florence, Biblioteca Medicea Laurenziana, Ms. Gaddi 36 (henceforth **L**; containing only chapters 12–15), is fundamentally different from the corresponding chapter in the other manuscripts. Internal evidence shows beyond doubt that it is older. As argued by Giusti, it is likely to represent the 1202 shape of the chapter (further arguments for this can be

[11] Thus my review [Høyrup 2021b: 622]. A page with documentation follows.

added).[12] As we shall see on p. 10, it is also possible that manuscript **V** conserves the 1202 version of the end of chapter 9 and the beginning of chapter 10 and took the reference to a Castilian master from there.

In the critical edition, Giusti [2020: lxxxiii] identifies a "series of omissions and errors that cannot reasonably be attributed to the author" and which are present in all the surviving manuscripts (except **L**, chapter 12). He therefore supposes that all descend from a single archetype ω which was already different from Fibonacci's original. However, comparison of the passages where Giusti corrects the text against ω with the "1202" version of chapter 12 in **L** (using also the critical commentary indicating where Giusti emendates the latter text) shows 71 agreements or near-agreements[13] of the latter with ω and 19 agreements[14] with the corrected text. In 32 cases,[15] the passage or section in question has no 1202 counterpart.[16] The only reasonable explanation is that Fibonacci produced the 1228 version on the basis of a master copy surviving since the inception

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[12] Since chapters 13–15 of **L** agree with the 1228 text known from other manuscripts it appears that its writer, after having finished chapter 12, got a manuscript of the later version. An additional note in chapter 12 drawn from the 1228 text corroborates that inference, cf. below, note 37.

[13] XII.§105m, XII.§144, XII.§173, XII.§173, XII.§173m, XII.§175, XII.§213, XII.§240, XII.§259, XII.§260, XII.§262, XII.§275, XII.§327, XII.§332, XII.§342, XII.§354, XII.§360, XII.§368, XII.§387, XII.§390, XII.§390, XII.§464, XII.§464, XII.§469, XII.§475, XII.§694, XII.§695, XII.§700, XII.§770, XII.§801, XII.§822, XII.§838, XII.§838, XII.§839, XII.§840, XII.§842, XII.§851, XII.§871, XII.§872, XII.§888, XII.§892, XII.§900, XII.§913, XII.§917, XII.§919, XII.§998, XII.§998, XII.§999, XII.§999, XII.§999, XII.§999, XII.§1015, XII.§1015, XII.§1024, XII.§1093, XII.§1126, XII.§1127, XII.§1128, XII.§1133, XII.§1137, XII.§1169, XII.§1173, XII.§1253, XII.§1270, XII.§1272, XII.§1272, XII.§1272m, XII.§1275, XII.§1279, XII.§1288, XII.§1318 – paragraph numbers from [Giusti 2020].

[14] XII.§105m, XII.§109, XII.§183, XII.§183m, XII.§187, XII.§282, XII.§369, XII.§766, XII.§775, XII.§813, XII.§813, XII.§813, XII.§814, XII.§897, XII.§903, XII.§911, XII.§930, XII.§1053, XII.§1318.

[15] XII.§34, XII.§149, XII.§200, XII.§293, XII.§305, XII.§338, XII.§339, XII.§339, XII.§482, XII.§491, XII.§493, XII.§499, XII.§502, XII.§560, XII.§572, XII.§618, XII.§619, XII.§619, XII.§632, XII.§637, XII.§647, XII.§712, XII.§716, XII.§720, XII.§721, XII.§721m, XII.§743m, XII.§743m, XII.§793, XII.§897, XII.§1074, XII.§1286. Some of these represent new material, others reformulation of passages.

[16] I leave out XII.§919~Gaddi:676, where inconsistent spellings in both (the only difference here) make comparison impossible.

of the *Liber abbaci* in 1202, with accumulated corrections and extensions, and a minor branching (often consisting of grammatical or other easy emendations) giving rise to the **L** version of chapter 12. Whoever has tried to return after three days to something one has written should know that the concept of errors that "cannot reasonably be attributed to the author" is more than slippery.

There is strong evidence that this keeping of a (possibly evolving) master copy from which further copies were made was Fibonacci's general habit; we find traces of it in the *Liber quadratorum* as well as in the *Flos*. First the *Liber quadratorum*. A single copy of the Latin text has survived [ed. Boncompagni 1862: 253–279], and a somewhat free vernacular version is found in the manuscript Florence, BNC, Palatino 577 [ed. Picutti 1979].[17] Benedetto da Firenze gives a philologically more conscientious vernacular version in his *Trattato di Praticha d'arismetrica*,[18] in which on fol. 501^{r} we read that *per insino a qui è scritto quanto allo illustro imperadore. Ora seguita lo scritto adiriçato a mess. R. cardinale*, "until here was written as to the illustrious Emperor. Now followed what was written to mess. R[ainero Capocci], cardinal". That is, Benedetto had access to two manuscripts, one derived from a version offered to the Emperor (Frederick II von Hohenstaufen), the other from one offered to Rainero Capocci – obviously made from the same master copy staying in Fibonacci's possession.

Rainero Capocci is also the dedicatee of *the prologue* to the *Flos* [ed. Boncompagni 1862: 227]. The text following after the prologue, however, is addressed to *vestra maiestate, gloriosissime princeps Frederice*, that is, once again to the Emperor. So, even from the master copy of the *Flos*, Fibonacci made several versions (perhaps, perhaps not, more or less *different* versions),

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[17] Picutti misidentifies the author or compiler of this manuscript as Benedetto da Firenze. A reference on fol. 11^{v}–12^{r} to the "10 demonstrations and solutions" which the writer has given "in the first chapter of the eighth part" of his *praticha d'arismetricha* excludes this and shows the instead that this *praticha* belonged to the same family as the two anonymous manuscripts Florence, BNV, Palatino 573 and Vatican, Ottobon. lat. 3304, without being by necessity identical with either – both state to be students of Domenico d'Agostino Cegia, known as *il Vaiaio*, and both draw on a shared model that may also have been used by others.

[18] Siena, Biblioteca Comunale L.IV.21.

dedicated (here leaving clear traces within the text) to different persons.[19]

Within the *Flos*, there is also a curious passage [ed. Boncompagni 1862: 234] where Fibonacci states to have solved a particular problem in three different ways *in libro vestro*, "in your book".[20] This could hardly be written if Fibonacci had not already given a copy of the *Liber abbaci* to the Emperor, and nothing makes us sure that this was not done before one was given to Michael Scot, normally counted as the "dedicatee" of the 1228 version. The precise words in the prologue [B1;G3] are

> *Scripsistis mihi domine mi et magister Michael Scotte, summe philosophe, ut librum de numero quem dudum composui vobis transcriberem,*
> You wrote me, my lord and master Michael Scot, supreme philosopher, that I should transcribe for you the book about numbers I wrote some time ago.

At least in classical Latin, *dudum*, "some time ago", would refer to relatively recent past, certainly not to something done 26 years ago. Rather than representing an instance of badly understood Latin[21] this might mean that Michael knew about a copy given to the Emperor (at whose court he was active) and has asked for a personal copy.[22]

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[19] We should be aware that in manuscript times, what is here spoken of as a "dedication" refers to the gift of a copy of the work – closer to what a present-day author writes on the title page of a book given to a friend than to the printed dedication.

[20] These solutions to the problem are already in **L** [ed. Giusti 2017: 176–183]; the passage therefore does not tell us which version is referred to.

[21] I checked three medieval-Latin dictionaries ([Blaise 1975], [Du Cange et al 1883] and [Niermeyer 1976]). None of them list *dudum*, as they would probably have done if the word had undergone a shift of meaning. My colleague Aksel Haaning also finds no traces of a changed meaning in his lexical notes on the word (personal communication).

[22] The existence of an earlier version offered to Frederick opens the possibility that the earlier version of chapter 12 which we find in **L** does not go back to the 1202 but to this late revision. *If so*, however, the major revision producing the 1228 version should postdate the creation of the version given to Frederick, which would then probably be close to the 1202 original.

Given the previously strained relation between the Emperor and Pisa, it seems likely that the gift was made not too long before Frederick's stay in Pisa in 1226. Frederick being born in 1194, there is no reason to speculate about encounters during Fibonacci's trading visits to Sicily prior to 1202. However, it seems from the words of the *Liber quadratorum* [ed. Boncompagni 1862: 253], Fibonacci's *Pratica geometrie* [ed. Boncompagni 1862: 1], is dedicated to "magister Dominicus" (possibly some Dominicus Hispanus, cf. [Boncompagni 1854: 98 n.]), who, even he, had asked for a copy of a work on that topic *iam dudum inceptum*, that is, a work which Fibonacci had started writing "some time" ago – hardly long ago, since it is still a work in progress.[23] In this case, too, as argued by Barnabas Hughes [2010], the early version had not been kept secret but entered circulation on its own – according to Hughes surviving in the apparently "incomplete" vernacular versions of the work (obviously, Dominicus must also have heard about it[24]).

All in all, as we see, Fibonacci's habit was to keep master copies of his major works, from which he made secondary copies to be offered as gifts. From the meticulous descriptions in [Giusti 2020: lxxvii–lxxx, lxxxvi–xciv] of those manuscripts of the *Liber abbaci* that contain chapter 1 it can be seen that several of them were based on two or more earlier manuscripts; we can therefore not (for instance from the presence or absence of the dedication to Michael Scot) order them in groups descending from the master copy at different moments.[25] Of some interest is, however, that

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> *intellexerim quod dignatur vestra sublimitas maiestas legere super librum quem composui de numero, et quod placet vobis audire aliquotiens subtilitates ad geometriam et numerum contingentes*
>
> I understood that your sublime Majesty had deigned to read in the book I put together about number, and that it pleased You to hear some subtleties regarding geometry and numbers

that Frederick might have asked for a copy of the book and received the gift somewhat prior to the encounter, which took place in July 1226 [Stürner 1992: II, 386*f*].

[23] Although the identity of Dominicus (whether "Hispanus" or not) is not clear, this gives support to the idea that rumours about Fibonacci's mathematics may have reached the Sicilian court before 1220 – and thus also that Frederick may have asked for a copy of the *Liber abbaci* before visiting Pisa in 1226.

[24] Dedicatory letters are certainly not to be taken automatically as plain truth. But it would be offensive to refer to "your book" if the Emperor had not been given that book, or to requests if they had not been made. Making such claims falsely could only be counterproductive for whatever aim Fibonacci may have had.

[25] Noteworthy is, however, that in some manuscripts the dedication to Michael Scot appears in the margin and not in the very beginning of the main text [Giusti 2020: xxix–xxi]. It would seem reasonable that Fibonacci inserted it in the margin of his

V, the manuscript referring to a "Castilian master" in the beginning of chapter 9, copies the last part of chapter 9 ([B130;G225] onward, and chapter 10 until [B137;G238]) from another manuscript than the archetype it shares with a larger group of manuscripts, which must have presented a lacuna here [Giusti 2020: xc]. Its scribe can therefore be supposed to have taken the information about the Castilian master from this different manuscript, which could descend from the 1202 stage of the master copy.

Since there are no significant divergences between this last part of chapter 9 in **V** (and of chapter 10 as far as it goes in this manuscript) and in the other manuscripts that have conserved the whole of chapters 9 and 10 (in indubitable 1228 version), we may guess that these chapters were not changed between 1202 and 1228.

The use of borrowed material

This is as far as we can get for the moment concerning the relation between the 1228 and the preceding (presumably 1202) version. For the next step of the argument we shall need to analyze the last problem from chapter 15 part 2 of the *Liber abbaci;* purportedly this part deals with geometric questions, which however is only partially true, and does not correspond to the last problem.

The problem in question [B405;G622] asks for three numbers (which we may call *a*, *b* and *c*) such that $^1/_2a = {}^1/_3b$, $^1/_4b = {}^1/_5c$, $a{\cdot}b{\cdot}c = a+b+c$. The solution proceeds by means of the false position $(a,b,c) = (8,12,15)$. Then $a+b+c = 35$, $a{\cdot}b{\cdot}c = 1440$. Therefore, the positions have to be reduced by a factor $\sqrt{^{35}/_{1440}} = \sqrt{^{7}/_{292}}$.

The *numerical* squares of *a*, *b*, and *c* are spoken of as tetragons (*tetragonus*). This Greek term is used regularly in the *Liber abbaci* about geometric squares (once, in a cistern problem, about a cube). Nowhere in the work except here does it refer to a square of a number. This probably means that Fibonacci here builds on a Greek, ultimately Byzantine source (which he might have encountered in Sicily as well as in Byzantium). We cannot be sure, however – a Latin translation from the Arabic *might* have used it for *murabbaʿ* (cf. [Souissi 1968: 173]).

master copy when producing a copy for Michael where it appeared in the beginning. Those manuscripts that give it in the beginning of the text may descend from the copy given to Michael but could also have been made by copyists who moved it to the fitting location.

A borrowing is anyhow obvious. The way it is done illustrates how Fibonacci deals with adopted material. The solution and discussion of the problem is organized in three sections. In the first, the term *tetragonus* is used 11 times, while *quadratus* is absent. In the second, which explains why a square root has to be taken, *quadratus* is used 5 times, *tetragonus* never; this is obviously an explanation added by Fibonacci, in which he uses his own language. In the last section, which verifies the outcome and which can be presumed also to be borrowed, *quadratus* is absent, while *tetragonus* is used 16 times.[26]

Fibonacci is thus highly faithful to the original when he borrows, but he does not emulate its style in added material or commentaries. He combines faithfulness with *deliberate* avoidance of imitation or pastiche – these would have induced him to carry over *tetragonus* to the commentary in the middle.

We should not conclude from this and other instances of faithful copying that Fibonacci was a compiler who did not understand what he put into his book. The explanation in the middle section, and copious parallels, show the contrary. Faithful copying was presumably a strategy making sure that no unintended misunderstandings creep in. We may think of an explanation offered by Charles Homer Haskins [1924: 152] of the tendency to translate Greek texts *de verbo ad verbum* (in part paraphrased from a 12th-century translator's preface). It had nothing to do with ignorance. Instead,

> Who was Aristippus that he should omit any of the sacred words of Plato? Better carry over a word like *didascalia* than run any chance of altering the meaning of Aristotle. Burgundio might even be in danger of heresy if he put anything of his own instead of the very words of Chrysostom.

As pointed out by Haskins, the translations he discusses are "so slavish that they are useful for establishing the Greek text".

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[26] Statistically, this distribution is highly significant irrespective of the model we use. The simple model that the probability to choose *tetragonus* is $^{27}/_{32}$ and that to choose *quadratus* is $^{5}/_{32}$) shows the probability of the present distribution to be slightly below 10^{-6}. A model based on combinatorics (taken for granted that *tetragonus* occurs 27 and *quadratus* 5 times and that the distribution over the 32 slots is random) gives a probability of $^{5!\times 27!}/_{32!}$, close to $5\cdot 10^{-6}$.

The lettering of diagrams

What we learned from the way this problem is dealt with should be taken into account when we address the lettering of geometric diagrams. Some diagrams carry the letter sequence *a-b-c-d–...*, others the sequence *a-b-g-d-...*; a few, continuing a discussion based on one of these sequences, starts with letters from later in the alphabet.[27] *a-b-c* obviously builds on the Latin (that is, Fibonacci's own) alphabet, while *a-b-g* could be inspired by Greek or Arabic material, including material translated from any of these languages into Latin (the translators never use *a-b-c*). Since no preceding Latin mathematical writer is known from whom Fibonacci could have borrowed *a-b-c*-diagrams, it seems plausible that such diagrams are of Fibonacci's own making. The avoidance of imitation and pastiche which we have observed would speak against the idea that *a-b-g*-diagrams (or some of them) were so too. Further, a writer who avoided as carefully as Fibonacci to refer to any sources beyond Euclid (and once Ptolemy together with what can be regarded as an explanatory commentary) would hardly try to intimate by his lettering of diagrams that his own inventions were borrowed.

This general conclusion is confirmed by analysis of Fibonacci's texts – not only the *Liber abbaci*.

We may start by looking at the diagrams in the beginning of Fibonacci's *Pratica geometrie* [ed. Boncompagni 1862: 2, 5*f*]. At first comes a diagram proving *Elements* I.28 (not identified but following a generic reference to Euclid); it is lettered *a-b-g-d-e-...* and is almost certainly borrowed from the version translated directly from the Greek [ed. Busard 1987: 42]. Somewhat later, when Fibonacci speaks about how to measure a "quadrilateral and equiangular field", an illustrating diagram is lettered *a-b-c-d-e-f-g;* when going on with more complicated divisions of the square, Fibonacci returns to *a-b-g-...* .

The *Liber quadratorum* is also illuminating. It begins by using the production of square numbers as sums of successive odd numbers starting

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[27] In a single case [B176;G299], the lettering seems to be *a-b-t-d...*, due to a misreading of *c* as *t* – in some manuscripts systematic, in others unsystematic. The two letters are indeed very similar and easily mistaken if one does not follow the mathematical argument. Since this is the very first lettered diagram in the work, copyists may not have been accustomed.

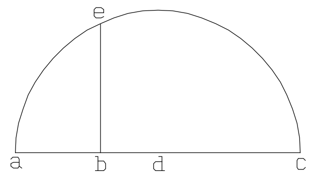
from 1 as a way to find Pythagorean triples (not Fibonacci's term). For example, 1+3+5+7+9 = (1+3+5+7)+9, whence $5^2 = 4^2+3^2$. This (which builds upon traditional "Boethian" familiarity with figurate numbers) is illustrated by a line diagram [ed. Boncompagni 1862: 254] lettered *a-b-c-d-e-f*. After that, almost all diagram letterings start *a-b-g* (when not using later alphabetic sequences because they continue an earlier argument using *a-b-g*); Fibonacci appears to have drawn the bulk of the work from an Arabic source – perhaps from a preceding Latin translation of an Arabic treatise. In the end of the treatise [ed. Boncompagni 1862: 279] comes "a question asked me by master Theodorus, philosopher of the Imperial Lord [Frederick II]":

> I want to find three numbers which collected together with the square on the first number make a square number. And if to this square the square of the second is added, comes out a square number; and when to this square the square of the third is added, similarly a square number comes from it.

Here, the letter sequence in the diagram used in the first transformation of the question is *a-b-c-d-e*. Then a lemma is proved, with the sequence *a-b-g-d-e-f-i*. The first transformation obviously had to be produced by Fibonacci for the specific purpose. But that created a situation where a lemma already at hand could be used. Once again this shows that Fibonacci borrowed, but borrowed with full understanding and ability to use creatively. Further *a-b-g* diagrams follow, coupled to text which provides numbers with the unit *dragma* and also makes use of ascending continued fractions, conclusive proof of a direct or indirect loan from an Arabic source.

In the *Liber abbaci*, *a-b-c* diagrams are as a rule used in simple situations where there was no need to borrow. The very first [B175;G299] is the one referred to in note 27. Like the commentary inserted within the *tetragonus*-problem it is used to explain why a square root has to be taken in a higher-degree problem solved by means of a single false position; it is absent from **L** where it should have been expected [ed. Giusti 2017: 34]. It is thus an explanation of a procedure which was already in the book; that makes it highly improbable that Fibonacci could have found it elsewhere. The introductory words, *quod etiam demonstrabo cum figura geometrica*, "which I shall also demonstrate by means of a geometric figure" point in the same direction.[28]

The following *a-b-c* diagram is in chapter 13 [B321;G503],[29] and is an alternative (*secundum alium modum*) to a previously given explanation of why the double false position works (the first, basic way is based on an *a-b-g* diagram); it could thus well Fibonacci's personal addition.



Next time the *a-b-c* sequence turns up [B353;G549] is when Fibonacci shows how to construct for example √10 geometrically. The style is simple and evidently Euclidean (related to II.14 as well as VI.13); but it is not taken from the *Elements*, neither from any of the Latin translations nor from the Greek text we now use.

And then an *a-b-c* diagram is used [B359;G557] to explain why 4√20 = √320 – *nam si ad oculum deprehendere vis quomodo*, "and if you want to discern by the eye how". Even this looks, from this initial phrase as well as from the context, to be a piece of personal pedagogy invented for the purpose.

For the while this must suffice to show the kind of situations where diagrams are lettered *a-b-c-*.... They correspond well to the promise from the prologue to add "a few things from my own mind". As we shall see, those situations where letterings *a-b-g-*... turn up are much more likely to reflect borrowed material.

What happened to chapter 12?

Apart from what was said on p. 10 concerning the end of chapter 9 and the beginning of chapter 10 (where nothing appears to have been removed or added), the only chapter where we can see directly what was changed in the 1228 version is chapter 12.

An important addition is the explanation of the *regula recta*,"direct rule", first-degree rhetorical algebra with unknown *res*, "thing". It had already been used abundantly in the 12th-century translation *Liber augmenti et*

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[28] With a single exception [B393;G603], the verbal form *demonstrabo* when coupled to a lettered diagram turns up together with the *a-b-c-*... sequence. In the exceptional case it might actually be a personal contribution to material otherwise taken over.

[29] The diagram is absent from Boncompagni's edition and thus from **F**, but his text refers to it.

diminutionis [ed. Libri 1838: 304–376][30] under the name *regula* and with unknown *census*,[31] as an alternative to the double false position. As shown by the double terminological difference, Fibonacci cannot have borrowed from there.

The *regula recta* turns up in an alterative solution to a "give-and-take" problem said to have been proposed by a master from Byzantium[32] [B190; G324]: a first man (A) asks from a second (B) 7 *denari*, saying that then he shall have five times as much as the second has. The second asks for 5 *denari*, and then he shall have seven times as much as the first. In the 1228 version, the first solution is supported by a line representation,

a e g d b

where *ab* represents the shared possession, *ag* the possession of A, and *gb* hence that of B. *gd* is 7, and *eg* 5. If B gives 7 to A, he shall be left with *db*, while A shall have *ad*. Therefore, if *ad* is divided into 5 parts, each of these shall equal *db*, for which reason *db* is $^1/_6$ of *ab*. Similarly, *ae* is $^1/_8$ of *ab*. That is, if $^1/_6+^1/_8$ of *ab* is removed, we are left with 5+7 = 12 – which is solved by "the rule of a tree".[33]

L instead gives a purely verbal argument running along the same lines. *This use* of line diagrams is absent from the 1202 version of the *Liber abbaci*;[34] maybe Fibonacci did not know it yet.

30 Transformed in a critical edition by Barnabas Hughes [2001].

31 The standard Toledo translation of Arabic *māl*, "possession", "amount of money", etc. Libri, in his mathematical commentary, identifies it with the *census* used in al-Khwārizmī's algebra (see below), and therefore in his mathematical commentary translates all the linear problems Et nos ponamus hec in figura, ut que dicere volumus clarius videantur occurring in the work as if they were dealing with x^2. His edition offers no serious problems.

32 Certainly as a challenge, cf. the "cut and thrust of disputation" referred to in the prologue.

33 That is, following the model of a problem about a tree [B173;G296], of which $^1/_3+^1/_4$, under ground, equal 21 palms. It is solved by means of a single false position (or rather, first by means of a "magisterial" rationalization of this method, then using the method in normal formulation) – details below, p. 60. In another tree problem the fraction under ground is given together with the absolute length above ground.

34 A different use with which he was certainly familiar turns up in chapter 14 part 3, almost certainly already in the 1202 version – see below.

In the alternative by *regula recta*, B is posited to possess a *thing* (*res*)[35] and 7 *denari*. Having received 7 *denari*, A therefore has 5 *things*, originally thus 5 *things* less 7 *denari*. If instead B gets 5 *denari* from A, he shall have a *thing* and 12 *denari*, while A shall have 5 *things* less 12 *denari*. Therefore, a *thing* and 12 *denari* equals 7 times 5 *things* less 12 *denari* – etc.

Fibonacci explains that the *regula recta* is used by the Arabs, and is very praiseworthy. This explanation, as well as the whole alternative procedure making use of it, is absent from **L**. The *regula recta* itself is not. A problem in the same chapter about selling pearls in Byzantium for bezants [B203; G340] and thus also likely to have been encountered in that place[36] is solved first by means of a single false position and next, alternatively, by means of the *regula recta* (*de eodem per regulam rectam*), and in this case both solutions are already in **L**. In 1202, we may conclude, when teaching what he had learned (as promised in the prologue), Fibonacci already knew the *regula recta* and applied it; in 1228 he then realized that an appurtenant commentary might be adequate, and inserted it at the first place where he now used the rule (which happens to be before the pearl-problem).[37]

The same structure turns up if we look at the occasional use of two

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35 Henceforth, to ease understanding, I shall italicize words that serve as unknowns when translating Fibonacci's rhetorical algebra (including the *regula recta* algebra). This artifice has no counterpart in the original.

36 When a particular problem was presented to Fibonacci by a Byzantine master [B188,190,249;G319,324,405], prices are mostly indicated in bezants – once only [B190; G324], unspecific "money" is spoken of; when problems refer to Byzantium [B161,203,274,296;G277,340,440,443], bezants are always used.

All problems that refer to bezants need not point to Byzantium, however. On one hand we cannot be sure that Fibonacci would always connect currency and locality in this way. On the other, bezants were also minted in Arabic and crusader countries; at times Fibonacci specifies when such variants are meant, as done twice on p. [B137;G238] (*hyperperi or Saracen*, and *garbi*, from the Arabic "West") – but that is no guarantee he would always do so. Bezants constitute circumstantial but not firm evidence.

37 However, after the *regula recta* solution to the pearl problem (in 1228 actually the first solution by means of that rule, cf. p. 60) Fibonacci explains that "there is another way which is called *regula versa*" (proceeding the opposite way). This has been inserted as a supplementary note in **L** (see [Giusti 2017: 79 app]), confirming that the writer of that manuscript got access to the 1228 text after having written chapter 12. Even he may have thought that an explanation was required.

algebraic unknowns.[38] The first instance [B212;G355] is in a purse-finding problem:

> Two men, who have *denari*, find a purse containing *denari*. When they have found it, the first says to the second, "if I get the *denari* in the purse together with the *denari* I have, then I shall have three times as much as you". Against which the other answered, "and if I get the *denari* of the purse together with my *denari*, I shall have four times as much as you".

At first two variants of a solution by means of two nested simple false positions are offered. Both of these are also in **L** [ed. Giusti 2017: 92*f*]. Then, in 1228 but not in **L**, a third way is proposed where the possession of the first man is posited to be a *thing* (*res*), and then this *thing* and the *purse* (*bursa*) are operated on as two rhetorical algebraic unknowns. The *regula recta* is not mentioned, but clearly in play.

More complicated variants (with more men, and sometimes several purses, and with similar cyclical conditions) follow in both versions, not all of them shared. Those that are are generally solved by means of unexplained numerical prescriptions, where the ratios are inserted into a scheme. One of them ([B216;G361], ed. [Giusti 2017: 98] deals with three men. In both versions it is solved in two ways, both based on a numerical prescription, about which it is said in **L** [ed. Giusti 2017 that

> where this rule come from and why one has to do like this we shall omit to demonstrate, as it is indeed very laborious. In what follows, however, we shall demonstrate something.

This has disappeared in the 1228 text – maybe because there Fibonacci gives reasons for the solution of some of the purse problems that are not shared. One about four men [B225;G372] may serve as example:

> The first and the second with the purse have the double of the *denarii* of the third; and the second and the third the triple of the fourth, and then the third and the fourth the quadruple of the first, while the fourth and the first with the purse similarly have the quintuple of the second. The solution to this problem you will find by finding the ratio of the *denarii* of the purse to the *denarii* of the first in this way. Because the first and

[38] One may claim that Fibonacci deals with several unknowns in many problems and submits them to operations very similar to what is done in equation algebra – cf. [Lüneburg 1993, *passim*] and [Sigler 2002: 626]. For the present purpose I shall stick to the restrictive usage of [Høyrup 2019] (close to the one introduced in [Heeffer 2010: 61]), which *inter alia* asks for consistent naming and explicit positing. The others we may call quasi-algebraic.

> second with the purse have the double of the third, half of the *denarii* of the first and second and the purse is as much as the *denarii* of the third man. Similarly from the other propositions you will have that $^1/_3$ of the second and third man and of the purse is as much as the *denarii* of the fourth man, and $^1/_4$ of the third and fourth man and of the purse is the quantity of the *denarii* of the first, and $^1/_5$ of the *denarii* of the fourth and first man and of the purse is the quantity of the *denarii* of the second. And because $^1/_2$ of the first and second and of the purse is the quantity of the third, the third part of the first and second and purse, that it $^1/_6$ of them, is $^1/_3$ of the third man. Commonly are joined $^1/_3$ of the *denarii* of the second and purse: then will $^1/_6$ of the first and $^1/_2$ of the second and of the purse be as much as $^1/_3$ of the second and third and of the purse. But $^1/_3$ of the second and third and of the purse is the quantity of the *denarii* of the fourth man; hence $^1/_6$ of the first and $^1/_2$ of the second and of the purse are the quantity of the *denarii* of the fourth man. Therefore $^1/_4$ of $^1/_6$ of the *denarii* of the first, that is, $^1/_{24}$, and $^1/_4$ of $^1/_2$, thus $^1/_8$ of the *denarii* of the second and of the purse, are $^1/_4$ of the *denarii* of the fourth man. Commonly are added $^1/_4$ of the third and of the purse: then $^1/_{24}$ of the first with $^1/_8$ of the second and with $^1/_4$ of the third and $^3/_8$ of the purse will be as much as $^1/_4$ of the *denarii* of the third and fourth and of the purse. But $^1/_4$ of the third man and the fourth and of the purse is the quantity of the first. Therefore $^1/_{24}$ of the first and $^1/_8$ of the second and $^1/_4$ of the third and $^3/_8$ of the purse are as much as the *denarii* of the first. Then their fifth part, that is $^1/_{120}$ of the first and $^1/_{40}$ of the second and $^1/_{20}$ of the third and $^3/_{40}$ purse, are $^1/_5$ of the *denarii* of the first. Commonly are added $^1/_5$ of the fourth man and the purse: then $^1/_{120}$ of the first and $^1/_{40}$ of the second and $^1/_{20}$ of the third and $^1/_5$ of the fourth and $^{11}/_{40}$ of the purse will be as much as $^1/_5$ of the fourth man and the first and of the purse. [...]

The final omission [...] is as long as the part that was translated. It leads to

> Hence $^{79}/_{600}$ and $^1/_{150}$ of the first, that is $^{83}/_{600}$ of the same, with $^1/_{25}$ of the purse, are $^{29}/_{200}$ of the purse. Commonly are taken away $^1/_{25}$ of the purse. Remain $^{83}/_{600}$ of the first, as much as $^{21}/_{200}$ of the purse. Then two numbers should be found so that $^{83}/_{600}$ of the first are $^{21}/_{200}$ of the second, they will be 63 and 83. Then if the first man has 63, the purse is 83. [...].

This is what was referred was spoken of in note 38 as a "quasi-algebraic" procedure. If only a single name had been used for "the *denari* of the first / first man", "the quantity of the *denari* of the first man", "the quantity of the first man" and "the first man" (and similarly for the others), this would have been rhetorical algebra with five unknowns: we observe the additions and subtractions performed "commonly", that is, from both sides of an equation, and the complicated substitutions.

For us it is not easy to follow the argument without making algebraic notes. There are traces in the text that even Fibonacci *described* a procedure performed by other means. Several errors are of the type that might occur when such a procedure is transferred: $^1/_{150}$ instead of $^1/_{150}$ *primi* and *denariis secondi* instead of *denariis primi*. Both are ω-errors, that is, they belong to Fibonacci's evolving master copy. So, when Fibonacci describes the procedure in rhetorical quasi-algebra he appears to copy from somewhere, and with high probability from his own calculation. This *could* be a solution by rhetorical algebra made separately, but it could also be (might rather be) an argument based on line diagrams similar to the one he used in 1228 in the first solution to the give-and-take problem. Once Fibonacci had learned that technique and used it for his own purposes and not in borrowed calculations he may have avoided to show it.

There are other quasi-algebraic solutions to purse-finding problems that are not shared with **L**. In the section about repeated travels with relative gain and absolute expenses ([B258;G417], [ed. Giusti 2017: 127] onward) we then discover that Fibonacci was familiar with the *regula recta* with two unknowns already in 1202, in spite of his seemingly *new* discovery of quasi-algebra with many unknowns (possibly based on an equally new discovery of corresponding line arguments) in 1228.

The first problem of this type is a recreational classic, "the merchant's nightmare" ([B258;G417]; [ed. Giusti 2017: 127]):

> Somebody proceeding to Lucca made double there, and disbursed 12 *denari*. Going out from there he went on to Florence; and made double there, and disbursed 12 *denari*. As he got back to Pisa, and doubled there, and disbursed 12 *denari*, nothing is said to remain for him. It is asked how much he had in the beginning.

Mostly, problems of this type had been solved by backwards calculation – after doubling in Pisa, the merchant had 12 *denari*, before doubling thus 6 *denari*, which he brought from Florence, etc. Fibonacci, in both versions, instead makes use of the principles of composite interest and discounting within a tacit single false position: an initial capital of 1 *denaro* would grow to 2·2·2 *denari* = 8 *denari*. The Lucca value of the expenses, instead, are (2·2+2+1)·12 *denari* = 84 *denari*. The initial capital must therefore be (2·2+2+1)·12/8 *denari*. Both offer an alternative by *regula recta* where the initial capital is posited to be a *thing*, confirming that Fibonacci knew this rule in 1202 but had not discovered the pedagogical need to introduce it.

Variations follow where, for instance, the expenses and not the initial capital are unknown, or where the rate of gain varies. They are all solved similarly (in both versions). Then, however ([B264;G426], [ed. Giusti 2017] comes a problem where this will not suffice:

> Again, in a first travel somebody made double; in the second, of two, three; in the third, of three, 4; in the fourth, of 4, 5. And in the first travel he expended I do not known how much; in the second, he expended 3 more than in the first; in the third, 2 more than in the second; in the fourth, 2 more than in the third; and it is said that in the end nothing remained for him. And let the expenditures and his capital be given in integers. We therefore posit by *regula recta* that his capital was an *amount* [*summa*], and the first expenditure a *thing*.

Now, indeed, the single false position would not suffice, and Fibonacci has recourse to *regula recta* with two unknowns, still without noticing that an introduction would be adequate for readers who have never heard about the method; what follows is a blameless algebraic calculation with two unknowns. Since Fibonacci in 1228 explains the *regula recta* to be an Arabic praiseworthy method (and since we know it to have been used before his time in the *Liber augmenti et diminutionis*), it is obviously a borrowing just like the use of line diagrams. The quasi-algebraic procedure, on the other hand, *could* be his own creation; in any case I have observed nothing in the text that contradicts it.

Chapter 13, the "rule of double false"

We shall return to a number of informative problems from chapter 12, but before that we shall have to look at chapters 13, 14 and 15, none of which is known in 1202 version.

Chapter 13 [B318–352;G499–545] is stated in its heading to deal with "*regula elchataym, qualiter per ipsam fere omnes questiones abbaci solvuntur*", "the *elchataym* rule, by which almost all abbacus questions can be solved", and it is explained in the very first line that *elchataym* is "an Arabic rule, in Latin translated as 'of two false positions',[39] by which [positions] the solution to almost all questions can be found". As it turns out, "all abbacus questions" (shortened "all questions" in the first period of the ensuing text) are such as are dealt with in chapter 12. It is pointed out that "tree problems" can be solved by means of a single false position.

[39] It is indeed the genitive dual *khata'āni* of *khaṭa'*, "error, mistake" – thus "of two errors".

Around a fifth of the chapter is dedicated to an introduction containing general explanations and proofs; the rest shows examples of the application of the rule, on one hand (part 1) to problems that have already been dealt with (namely in chapter 12), on the other (part 2) to problems that so far have not been presented.

We shall return to one of these problems, and for the moment look at the introduction. At first two interpretations of the rule is explained. One, in present-day terms, is a linear interpolation between or extrapolation from the outcome of two guesses. Fibonacci uses this example:[40] 100 *rotuli*[41] are worth 260 *soldi*, what is 1 *rotulo* worth? A first position is that it is worth 1 *soldo, then 100* rotuli would be worth 100 *soldi*, 160 too low; a second position is that it is worth 2 *soldi*, then 100 *rotuli* would be worth 200 *soldi*, 60 too low. By the Rule of Three, the position has to be increased by $^{(1\cdot 60)}/_{100} = {}^{3}/_{5}$ *soldo*, that is, to $2^{3}/_{5}$ *soldo*.

The other method (the one which normally goes under the name of "double false") "mixes" the two guesses in such a way that the errors cancel: if we take the first guess 60 times, we get a deficit of 60·160 *soldi*, while taking the second guess 160 times gives a deficit of 160·60 *soldi*. Subtracting the former from the latter (that is, making the position 1 *soldo* 60 times and the position 2 *soldi* 160 times) cancels the error; but we should only makes a single position, whence the true result is $^{(160\cdot 2-60\cdot 1)}/_{(160-60)}$ *soldi*.

All this (with variant examples where one position leads to an excess and the other to a deficit, or both to an excess), including integrated arguments of proportionality for the validity of the first method, makes use of marginal schemes similar to those used in the chapters explaining the Rule of Three (which Fibonacci never gives a name), barter, partnership and alloying. In these, numbers are written inside a rectangular frame, probably reflecting the *lawha* (clayboard) within which such calculations were made by Maghreb reckoners.[42] They are likely to go back to the 1202 version.

Then [B320;G502], *ut unde hec provenient demonstrentur*, "so that it may be demonstrated where this comes from", follows proofs of the second

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[40] Simplified – Fibonacci moves on two metrological and three monetary levels.

[41] The *rotulo* is a weight unit, corresponding to the Arabic *raṭl*, depending on location between one and 2½ pounds.

[42] See [Lamrabet 1994: 203] and [Abdeljaouad 2005: 36*f*].

method, based on line diagrams lettered *a-b-g-d-e-z-i* – different according to whether both errors are in excess, both in deficit, or one in excess, the other in deficit. The change of style indicates that this is has been added in the 1228 version.

We have no indications of the source beyond the letter sequence, which could be Greek or Arabic, and the total absence of anything similar from what is known from Byzantium.

From the Arabic world I also do not know anything quite similar. On the other hand, the earliest extant description of the method in Arabic, that of Qustā ibn Lūqā (second half of the ninth century, ed. trans. [Suter 1908]), contains geometric proofs. These are based on rectangles (quite similar to what *we* would do in a coordinate system), but a transformation of these into proof like those of Fibonacci would correspond to the transformation of the propositions of *Elements* II into their "key" version (see below, p. 23). Moreover, the manuscript used by Suter was copied in India in 1722; Qustā's ideas thus circulated well and may have inspired broadly.

Nothing, on the other hand, suggests a link to the way Maghreb authors like Ibn al-Bannā' [ed. trans. Souissi 1969: 88] and al-Qalasādī [ed. trans. Souissi 1988: 68] present the rule by means of "scale pans" – neither in Fibonacci's first explanation nor in this seeming addition. Yet there is no guarantee that the traders whom Fibonacci encountered as a young travelling merchant had read the precursors of Ibn al-Bannā' and al-Qalasādī (both postdate Fibonacci), so he may have learned the rule itself from them (if so presumably adding the arithmetical explanations himself).

In the problems following after the introduction, all marginal material is of the kind known from the early chapters. Nothing indicates that anything was added in 1228, but there are clear signs that one problem was redacted (below, p. 36).

The composite nature of chapter 14

Chapter 14 exhibits a number of internal ruptures betraying (imperfect) revision.

Much of chapter 14 builds on *Elements* X. In the *Flos* [ed. Boncompagni 1862: 228], Fibonacci states to have been inspired by a cubic problem presented by Giovanni di Palermo to study *Elements* X "more accurately" (*accuratius studui*), and that, "because it is more difficult than those books that precede or come afterwards, I began to gloss upon this same Tenth Book, reducing its understanding to number, which in itself is proved by lines and surfaces".[43] Since this happened at the occasion of Frederick's stay in Pisa in 1226, we might expect the 1202 version to contain perhaps some material related to *Elements* X but the 1228 version to contain much more.

This is amply confirmed by the text. We shall first have a look at the preamble to the chapter [B352;G547]:

> Let it be me permitted to insert in this chapter about roots certain necessary matters, which are called keys [*claves dicuntur*]; since they are all proved by clear demonstrations in Euclid's Second, it will suffice beyond their definitions to proceed by means of numbers. The first of which is that, when a number is divided into any number of parts, then the multiplications of these parts in the whole divided number, joined together, equal to the square of the divided number, that is, the multiplication of the same number in itself.[44] For example: let 10 be divided into 2, and 3, and 5. I say that the multiplications of the two, the three, and the five in 10, evidently 20, and 30, and 50, equal the multiplication of 10 in itself, that is, 100. [Similar versions of *Elements* II.1 and II.4 follow, together with the corollary $2a{\cdot}(a+b)+b^2 = a^2+(a+b)^2$]. Next, if a number is divided into two equal parts, and also into unequal parts, then the multiplication of the smaller part by the larger, together with the square of the number which there is from the smaller part until the half of the whole divided number will be equal to the square of the said half [*Elements* II.5; a numerical example and a similar version of II.6 follow]. To the latter two definitions are reduced all questions from *aliebra* et *almuchabala*, that is, in the book of *contemptio*[45] and *solidatio*.
>
> Then, finished this, this chapter is divided into five parts. Of which the first is about the finding of roots; the second about their multiplication in each other, and that of binomials. The third about their addition. The fourth about their mutual detraction. The fifth about the division of roots and of binomials.

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43 In the mid-15th-century manuscript Florence, BNC, Palatino 573, fol. 433^{v}–434^{r} this has developed into a claim that Fibonacci wrote "a book about the 10th of Euclid"; since Fibonacci elsewhere refers to "books" of his when we know they existed, we may take from his words that the 15th-century admirer extrapolated, and that Fibonacci did not go beyond glossing.

44 *Elements* II.1, applied to two equal lines; or, alternatively, *Elements* II.2 generalized to division into several parts.

45 As pointed out by Enrico Narducci [1858: I, 23], the word however spelled in the various manuscripts will be a mistake for *contentio*, "comparison / contrast / struggle".

We shall return to the presentation of the "keys" but first observe that the final listing of the parts contained in the chapter elucidates its fractured structure. It contains indeed seven, not five parts, two designated "second" and two designated "fourth" – I shall refer to them as "2a", "2b", "4a" and "4b":

1 *Prima quarti decimi capituli.*

2a *Pars secunda quartidecimi capituli de multiplicatione radicuum et de binomiorum.*

2b *Pars secunda de multiplicatione radicum in radicibus et numeris.*

3 *Tertia de additione et extractione radicum inter se et reliquorum duorum simplicium numerorum.*

4a *Pars quarta, de divisione trium simplicium numerorum inter se.*

4b *Pars quarta, de divisione binomiorum et recisorum per numeros ratiocinatos et inratiocinatos et econtra.*

5 *Pars quinta de inventione radicum cubicarum et de additione et multiplicatione et extractione seu divisione earundem.*

in translation,

1 First part of the 14th chapter

2a Second part of the 14th chapter about the multiplication of roots and of binomials.

2b Second part about the multiplication of roots by roots and numbers.

3 Third about mutual addition and detraction of roots and the other two simple numbers [that is, binomials and apotomes].

4a Fourth part, about the mutual division of the three simple numbers.

4b Fourth part, about the division of binomials and apotomes by rational and irrational numbers, and vice versa.

5 Fifth part about the finding of cube roots and about the addition, multiplication, and detraction and division of the same.

The heading for 2a is present in **F** but absent from **V**. Giusti's apparatus does not inform about the other manuscripts (nor about these two, in fact), but it is announced in the end of the preamble. It was thus intended by Fibonacci, who may at first have forgotten to write it; alternatively, some copyists have left it out because there is another heading for a "part 2". The former possibility seems most plausible; there is a jump in topic (from the extraction of the square roots of multi-digit numbers to the subject-matter of *Elements* X), so *some* heading would be required. A copyist not liking the full "second part of the 14th chapter about the multiplication of roots and of binomials" would therefore have cut it down, perhaps to "about the multiplication of roots and of binomials", and not have eliminated it altogether.

The heading for 4b is present in all manuscripts, but Giusti leaves it out from his reconstructed text, once again because there already is heading for a "part 4". The presence of the heading for part 2b leaves no doubt that it was already in Fibonacci's master copy, and thus that 2a was inserted during the revision; as we shall see when looking at the contents, 4a was included at the same occasion.

Before continuing in this direction, however, we shall return to the initial, major section of the preamble. As we see, it presents a numerical translation of as much of *Elements* II as needed in order to justify / replace the operations of second-degree algebra, speaking of the resulting propositions as "keys". The use of the present tense passive *dicuntur* shows that they are *already* called so, that is, the term is not invented by Fibonacci. Nor is this usage known from Arabic scholarly writing – there, for example in al-Kāšī's *Miftāḥ al-ḥisab*, the "key" is that which unlocks a subject (*in casu* calculation, *ḥisāb*). The reference to *aliebra almuchabala*, on the other hand, points to an Arabic origin.

The unusual Latin explanation "*contentio* and *solidatio*", provides us with the beginning of an explanation. It differs from the translation given by Fibonacci himself in chapter 15, and must therefore have been present in his source – which must hence have been Latin. That is: *The source for this notion and collection of "keys" must be a Latin translation of a work written at first in Arabic*.

Where could this translation have been made? We may think of Sicily or the Christian-Iberian ambience (Toledo, but not only), and can exclude neither, not least in view of the migration of scholars from one place to the other – for one, Michael Scot learned Arabic and started translating in Toledo before moving to Italy and to Frederick's court [Schramm 2001: 295].[46] The original on which this translation was based, however, was almost certainly produced in al-Andalus (Muslim Spain). That we know about nothing similar to the "keys" from surviving Arabic mathematics

[46] However, a misunderstanding has to be cleared away: According to [Kaunzner 1985: 8], in the lettering of diagrams *z* is replaced by ζ or ξ in certain Tuscan 15th-century manuscripts; he assume this to go back to Greek scholars working at Frederick's court. Inspection shows, however, that the letter in question is simply *ç*, often used (also in the running text of these manuscripts) instead of *z* – for example, in *terço, avança* and *sança* (*terzo, avanza* and *senza* in modern Italian).

may itself be a weak argument. It is somewhat strengthened by the observation that 12th-century al-Andalus scholarship was already largely isolated from the Arabic mainstream, as shown by the fate of ibn Rušd's (Averroës's) philosophy – much more influential indeed in Latin and Hebrew than in later Arabic philosophy.[47] To this comes the use of the keys (not called so, it is true) in the *Liber mahameleth*, produced in al-Andalus and more or less freely translated into Latin in the Toledo environment somewhere around 1260, perhaps by Gundisalvi.[48]

Part 1, teaching the numerical extraction of square roots, almost certainly goes back to 1202; we cannot exclude that some changes were made, but no internal ruptures suggest so; even the geometric diagram used to construct $\sqrt{10}$ geometrically fits in seamlessly. Of some interest is that the finding of $\sqrt{743}$ as $27^{7}/_{27}$ (first finding $729 = 27^2$ as the nearest smaller perfect square, and then using the first approximation $27+^{(743-729)}/_{(2\cdot 27)}$) is spoken of [B353;G549] as being "according to abbacus teaching" (*secundum abbaci material*). This seems to exclude that "abbacus" was intended simply to refer to "calculation, and that Fibonacci had something more specific in mind – perhaps something like the teaching type he encountered in Bejaïa, which would justify Burnett's translation of *studio abbaci* as "abbacus school" (above, note 3).

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47 According to [Samsó 2007: 134], even al-Bitrūjī's astronomy may only have been known in the Islamic East through Maimonides.

48 That is was produced in an Arabic environment is persuasively argued by Jacques Sesiano [2014: xviii]; the arguments even show that the original language was Arabic, not Latin. Sesiano's tentative identification of the author with Johannes Hispalensis ("from Seville) – namely because an indirect 15th-century quotation refers to him as *hispanus* – can probably be disregarded, in part because of counter-arguments advanced by Sesiano himself on the next page, in part (more decisively) because the "Toledan Regole" [ed. Burnett, Zhao & Lampe 2007] written by John has a wholly different algebraic terminology than the *Liber mahameleth* – see [Høyrup 2021a: 42–49], where the intricacies of the question are discussed and the use of the keys documented.

A reference in the text to what is normally done by "the Arabs" does not disprove an original Arabic authorship. It agrees perfectly with Gundisalvi's free translation style – in his translation of al-Fārābī's *Catalogue of the Sciences* [ed. Gonzales Palencia 1953: 98] we thus find the explanation that "the science of depths and heights, or of finding distances and many other things of this kind, are contained in full in a book that is with the Arabs".

Part 2A introduces the reader to the realm of *Elements* X. The initial terminology betrays familiarity with the translation made directly from the Greek (*riti* for Greek ῥητός, *potentia* for Greek δυνάμει). This is nothing new, Fibonacci's occasional use of this version has been discussed by Busard [1987: 18*f*] as well as Folkerts [2004: 109*f*]. After this beginning, Fibonacci goes his own ways – his aim is, after all, the arithmetic of roots and numbers, not geometry. So, he reformulates the definitions of the classes of binomials and apotomes – the former (now spoken of as *numeri*, "numbers") being explained and exemplified in this way:

1st: $4+\sqrt{7}$, $4^2-7 = 9$
2nd: $\sqrt{112}+7$, $112-7^2 = 63$, $63:112 = 3^2:4^2$
3rd: $\sqrt{112}+\sqrt{84}$, $112-84 = 28$, $28:112 = 1^2:4^2$
4th: $4+\sqrt{10}$, $4^2-10 = 6$, $6:4^2$ not as square number to square number
5th: $\sqrt{20}+3$, $20-3^2 = 11$, $20:11$ not as square number to square number
6th: $\sqrt{20}+\sqrt{8}$, $20-8 = 12$, $12:20$ not as square number to square number

Before offering a parallel but short presentation of the apotomes Fibonacci says [B357;G554] that the square of any of the binomials is a "first binomial" – actually stated the other way round, that the square root of a first binomial is a binomial, the counterpart of *Elements* X.60. He continues with the square roots of the remaining binomials, also counterparts of propositions from *Elements* X but referring to classes of irrationals that he has not defined yet – showing that he draws on some larger treatise but only piecemeal (and not editing too well). The proofs are extremely sketchy and purely arithmetical. As we shall see when discussing part 3, these proofs present us with further evidence that part 2a is a secondary insertion.

Binomial and apotome are designated *binomium* respectively *recisum* by Fibonacci. *Elements* X translated directly from the Greek [ed. Busard 1987] uses *ex duobus nominibus* respectively *abscisio;* the *Elements* translated by Gerard of Cremona [ed. Busard 1984], those translated by Adelard [ed. Busard 1983], Robert of Chester's redaction [ed. Busard & Folkerts 1992], that of Hermann of Carinthia [ed. Busard 1967] and the anonymous translation of an Arabic commentary to *Elements* X [ed. Busard 1997] all use *binomium* and *residuum*. None of these can have inspired Fibonacci. Occam's razor might make us postulate that the source for the major part of the preamble and that for the interpretation of *Elements* X must be the same, but this famous tool is, after all, not much more than the naive illusion that we already know all entities there are in the world.

Part 2b returns to what will naturally have belonged to the work already before Fibonacci started his closer study of *Elements* X in 1226. It begins by showing that $\sqrt{10}\cdot\sqrt{20} = \sqrt{(10\cdot 20)}$[49] and similarly, then how (for instance) $\sqrt{40}\cdot\sqrt{90}$ can be reduced. Next, as mentioned above, an *a-b-c* diagram is used [B359;G557] to explain why $4\sqrt{20} = \sqrt{320}$. Fibonacci also takes up multiplications involving roots of roots. As in part 1, no internal fractures suggest augmentation or removal.

Part 3 continues the material from 1202, stating initially that the sum of a number and a surd root can be nothing but a binomial (*binomium*) (illustrating this with an *a-b-c* diagram). Later it refers also to apotomes (*recisum*), and even to "first" binomials and apotomes, without having defined them. It appears that the pattern we know from the *regula recta* is repeated: in 1202 Fibonacci had referred to these classes as something known, and in 1228 he discovers that an explanation would be appropriate (inserting it as part 2a).

Further evidence that this is what really happened is a new proof [B362; G560] that the square on any binomial is a first binomial; as we remember, this was also (very sketchily) shown in part 2a, and there is no backward reference. The argument in part 2a was based on arithmetical considerations and (paradigmatic) examples. This time, the argument is geometric, making use of a diagram lettered *a-b-g-d-z-*... There are more *a-b-g-*... diagrams, enough to make it implausible that the demonstration just mentioned should be a late intruder. Already before making the 1228 revisions, Fibonacci must have used one or more written sources derived from *Elements* X.

The addition $4+\sqrt{\sqrt{10}}$ is performed in two ways [B364;G563]. "In the vernacular way" (*per vulgarem modum*) it is found by numerical approximation, as slightly less than $1^4/_5$.[50]. For the case that "you want

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49 The argument is based on letters *a-b-g-d* that are *not* carried by lines, as the Euclidean way would be. This is unique in the *Liber abbaci* and, I believe, also at the time in general. It is evidently copied – Fibonacci, working on his own, would not have chosen this sequence); but the source may well have used letter-carrying line segments.

50 Fibonacci does not explain, but a possibility would be first to approximate $\sqrt{10}$ as $3^1/_6$ (the habitual first approximation). Now, $3^1/_6 = 4\text{-}^5/_6$, and approximating from above we find $\sqrt{3^1/_6} \approx 2\text{–}(^5/_6)/(2\cdot 2) = 2\text{–}^5/_{24}$, which is indeed slightly less than $2\text{–}^5/_{25} = 2\text{–}^1/_5 = 1^4/_5$.

do it *magistraliter*" (a concept to which we shall return on p. 59) Fibonacci introduces a line diagram lettered *a-b-c* serving name-giving in the calculation $(4+\sqrt{\sqrt{10}})^2 = 16+\sqrt{10}+8\cdot\sqrt{\sqrt{10}}$ (which again leads to numerical approximation). Such higher binomials as fall outside the scope of *Elements* X were apparently also not dealt with in the source Fibonacci drew on here; in any case, he works on his own on the topic.

Part 4a, purportedly "about the mutual division of the three simple numbers" (rational numbers, roots, and roots of roots), actually deals with more.

A first section (roughly one fourth of the whole) does indeed treat of the mutual division of such monomials – beginning [B365;G565] with $\sqrt{30}/\sqrt{10}$. This may very well go back to 1202. Then comes a passage [B368; G567]

> When thus multiplications and additions and the extractions and divisions of simple numbers have been explained, that is, of those that can be represented by simple lines, and when also the multiplications of the three binomials by themselves have been shown, now should be shown how the roots of the fourth and fifth and sixth binomials are to be multiplied.

This is followed by the statement that

> The root of a fourth binomial is composed of two lines, of which one is the root of a fourth binomial, and the other is the root of the apotome having the same name. Of which lines the first is called a major, the second a minor, and the conjunction from them, that is, the root of the fourth binomial, is similarly a major; and it is called a major because the major name it has as power is a number.

The final explanation of the name "major" is evidently a folk etymology – and even mathematically wrong: as Fibonacci has shown, the square on any binomial is a first binomial and therefore fulfils the condition. The correct definition (no explanation of the name[51]) is given in *Elements* X.39. What precedes is correct. Translated into numbers, Euclid's definitions of "major" and "minor" (*Elements* X.39 and X.76) state that $M = \sqrt{(a+\sqrt{b})}$ is a major and $m = \sqrt{(a-\sqrt{b})}$ the corresponding minor if a^2-b is no square number.

Corresponding statements about the roots of a fifth and a sixth binomial follow – rather opaque and not evidence that Fibonacci understood. Then,

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[51] A plausible explanation of the origin of the name in the geometry of the regular pentagon was proposed by Marinus Taisbak [1996].

however, he gives a line demonstration lettered *a-b-g-...*, based on the example $M = \sqrt{(4+\sqrt{6})}$, $m = \sqrt{(4-\sqrt{6})}$, without explaining that (and *a fortiori* why) *M* and *m* are a major and a minor. So, Fibonacci probably understood, but made a rather incomprehensible extract from his source (as already said, his use of *Elements* X in the *Flos* shows that he actually understood it much better than one might believe from what is done here).

More of the same kind follows, still based on *a-b-g-...* lettered line diagrams. In middle [B370;G571] a demonstration of the sign rule "less times less makes plus" – more precisely, *multiplicatio rerum diminutarum crescenda sit*, "the multiplication of diminished things should be made increasing" is argued on a rectangle diagram lettered *a-b-c-d-...*: Another case of Fibonacci inserting in faithfully borrowed material an explanation of his own formulated in his own style.

From [B371;G572] onward the style changes, and the multiplication of binomials (also binomials involving roots of roots) with their corresponding apotomes is shown in number schemes similar to those used early in the work, for instance when the Rule of Three is applied. The shift is smooth – after having found the product of a binomial with its apotome by means of an *a-b-g-d* line diagram Fibonacci goes on, "and in order to show this in numbers". Most likely, this is part of what was added in 1228, but a pedagogical clarification of Fibonacci's own making.

Part 4b is simpler in structure. The first two-third corresponds to the heading, dealing with "the division of binomials and apotomes by rational and irrational numbers, and vice versa" – with the provision that roots of roots are included. The reduction of a division by a Euclidean binomial is explained on the basis of an *a-b-g*-diagram, that by a trinomial (exemplified by $10/(2+\sqrt{3}+\sqrt{5})$ with reference to an *a-b-c*-diagram, suggesting that the latter topic was added by Fibonacci himself to what he had borrowed. Nothing in the text forces us to assume this part was added in 1228; reversely, nothing excludes it.

Part 5 corresponds to its heading, "about the finding of cube roots and about the addition, multiplication, and detraction and division of the same". According to the use of *a-b-g*-lettered line diagrams, most of the material is borrowed; alternative ways to determine $\sqrt[3]{32}+\sqrt[3]{4}$ and $\sqrt[3]{32}+\sqrt[3]{4}$ are supported by an *a-b-c* diagram and are thus almost certainly added by Fibonacci.

Chapter 15, part 1

Chapter 15 is composed of three parts not connected by thematic continuity nor by cross-references but only held together by this short heading and preamble (both somewhat misleading) [B387;G595]:

> Begins the fifteenth chapter concerning rules of geometry,
> and questions of *algebra* and *almuchabala*
>
> The parts of this last chapter are three; of which the first will be about the proportions of three or four quantities, to which many solutions to geometric questions lead back; the second will be about the solution of certain geometric questions; the third will be about the way of *algebra* and *almuchabala*.

As we shall see (p. 34), there may be a reason that the heading seems to speak of two matters, and the preamble of three. But before pursuing this hint, we shall need to consider part 1.

Undeniably, many geometric questions are solved by means of proportions in the Euclidean tradition. However, very little in part 1 has to do with the service which proportions offer in geometry. I have analyzed it in [Høyrup 2011: 89–92] and in more depth in [Høyrup 2021a: 49–57] and shall only summarize here (using the same division into sections[52]).

Apart from a few general observations, all these sections are propositions stating that if, in a proportion, some terms are known together with certain sums of or differences between terms, then the remaining terms are known. Those involving three terms make use of lettered line diagrams, those involving four of segments labelled by letters (or, formulated the other way round, letters carried by segments). Those that are of the first degree are solved by means of proportion manipulations (transformations *permutatim*, *conjunctim*, *disjunctim*, etc., and the "product rule" that $a : b = c : d$ implies $ad = bc$); those that are of the second degree further apply *Elements* II.5 and 6 in "key" version (mentioning neither Euclid not the notions of "keys").

Sections #1–#3 deal with three numbers is continued proportion, with one term given together with the sum of the other two. A section "3a" follows which I did not count separately in my earlier discussions; it states that a number can be divided in infinitely many ways into a sum of three

52 Since both of these were written before the appearance of [Giusti 2020], they refer to the pagination in [Boncompagni 1857].

numbers in continued proportion. All employ the letter sequence *a-b-c*, which already suggests them to have been produced by Fibonacci himself as an introduction to what follows.

Sections #4–#5 still deal with numbers in continued proportion, but now one term and the difference between the two others are given. The letter order changes to *a-b-g*, but the subsequent arguments make use of *c*, indicating that Fibonacci borrowed but made calculations of his own.

After this beginning follows a large genuine piece of coherent theory, or almost [B389–395;G595–606]. It is inspired by the ancient theory of means – arithmetical, geometric, harmonic (the three known to Archytas), the three subcontraries to the latter two, and five more – see [Heath 1921: I, 87]. With one omission they are listed by Nicomachos (*De institutione arithmetica* II.xii–xxviii, ed. [Hoche 1866: 122–144], trans. [d'Ooge 1926: 266–284]), and with a different omission by Pappos (*Collectio* III.xii–xxiii, ed. trans. [Hultsch 1876: I, 70–105]). The following scheme shows them (adding an extra line at bottom, on which imminently) together with the corresponding sections of *Liber abbaci* part 15.1.[53] Everywhere, it is assumed that $P \leq Q \leq R$, and only the last line does not presuppose that $P < Q < R$. That is indeed the condition that Q can sensibly be considered a mean; it is therefore not strange that this last case is not included by the ancient writers. But they probably knew about it and why it had to be left out. Theon of Smyrna (*Expositio* II.lx [ed. trans. Dupuis 1892: 191] tells indeed that "the Pythagoreans have addressed these six means and their subcontraries broadly.[54]

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[53] In the *Liber abbaci*, the numbers are represented on this line diagram:

<u>*a* *d* *g* *b*</u>

where *ad* corresponds to P, *ag* to Q and *ab* to R.

[54] Heath's formulation [1921: I. 87],

> The two lists [of Nicomachos and Pappos] together give *five* means in addition to the first six which are common to both; there would be six more (as Theon of Smyrna says) were it not that $\frac{a-c}{b-c} = \frac{a}{b}$ is illusory, since it gives $a = b$

might give the expression that all of this was said by Theon, which is not the case. But it seems plausible that the observation was made in Antiquity.

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P} = \frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P} = \frac{R}{Q}$ or $\frac{R-Q}{Q-P} = \frac{Q}{P}$	P2	N2	27–29
$\frac{R-Q}{Q-P} = \frac{R}{P}$	P3	N3	7–9
$\frac{R-Q}{Q-P} = \frac{P}{R}$	P4	N4 (but inverted)	10–12 (inverted)
$\frac{R-Q}{Q-P} = \frac{P}{Q}$	P5	N52 (but inverted)	#34–36 (inverted)
$\frac{R-Q}{Q-P} = \frac{Q}{R}$	P6	N6 (but inverted)	#20–22 (inverted)
$\frac{R-P}{Q-P} = \frac{R}{P}$	absent	N7	#16–18
$\frac{R-P}{R-Q} = \frac{R}{P}$	P9	N8	#13–15
$\frac{R-P}{Q-P} = \frac{Q}{P}$	P10	N9	#30–32
$\frac{R-P}{R-Q} = \frac{Q}{P}$	P7	N10	#37–38
$\frac{R-P}{R-Q} = \frac{R}{Q}$	P8	absent	#23–25
$\frac{R}{Q} = \frac{R-P}{Q-P}$	absent	absent	#26

For all means except the trivial arithmetical mean (whose definition in terms of a proportion is inane), Fibonacci finds any of the three terms involved from the two other terms. In #26 he shows that it leads to $\frac{R-Q}{Q} = \frac{R-P}{Q-P}$, which, unless *P* be zero (*zephirum, hoc est nihil*), implies that *Q* = *R*. He does not observe that #27–#29 deal with a continued proportion, not that this is a systematic exploration; nor, *a fortiori*, does he say that it explores the ancient theory of means with its natural extensions (a theory which he may have known from Boethius's faithful translation of Nicomachos[55]). It seems obvious that he has borrowed the whole sequence, as already suggested by the letterings of diagrams, without thinking about its background.

[55] It is noteworthy, however, that Fibonacci never mentions Boethius (nor Nicomachos).

Sections #39–#50 treat of four-member proportions $\frac{P}{Q} = \frac{R}{S}$, using the same tools. It first shows how any member can be found if the three others are known, then how to proceed if two members are known together with the sum of or difference between the other two. #45 illustrates it with a commercial example involving *rotuli* and *bizantii*. This is of particular interest, since similar problems abound in the *Liber mahameleth*.[56]

Even #50 is interesting, and for two reasons. Here, beyond the proportion, P^2+Q^2, R and S are given. It makes use of the principle that if four numbers are in proportion, so are their squares. This is the contents of #6, which is wholly out of place where it stands; but there is neither forward reference there nor backward reference in #50, which shows that part 1 is not planned as a single sub-treatise but a compilation, from #7 onward (probably from #4 onward) borrowed, presumably not from a single source but rather from closely related sources. The reverse principle (that if numbers are in proportion, so are their square roots) is used in the *Liber mahameleth* in an analogous problem [ed. Sesiano 2014: 194].

Except for the few sections dealing with continued proportions and #39, this has nothing directly identifiable to do with geometric questions (and that little has been dealt with elsewhere in the book). It is understandable, on the other hand, that Fibonacci when coming across it thought it should be included in his "greater book about numbers" (as he refers to it in prologue to the *Flos* [ed. Boncompagni 1862: 227]), and also that he thought it belonged in the same chapter as the algebra. That appears to be what caused that the two topics of the heading of the chapter (probably of 1202 date) was expanded into three topics in the preamble; reversely, the discord between the heading and the preamble supports the assumption that part 1 was added in 1228.

Since nothing similar is known from other parts of the Arabic world, nor from Byzantium, the similarities with the *Liber mahameleth* point to al-Andalus as the ultimate origin – but quite likely used (like the source for the preamble to chapter 14) through an already existing Latin translation.

[56] For instance [ed., trans. Sesiano 2014: 234, 788*f*]:

> Three modii are given for thirteen nummi. What is the quantity of modii, bought at the price set, which, when added to their price, makes sixty?

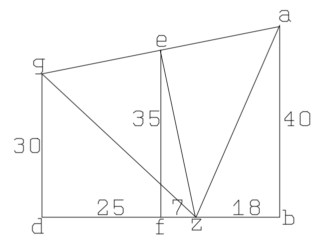
Chapter 15 part 2

Part 2 is announced as dealing with “the solution of certain geometric questions”. So it does, but not only. At some points, it is not the question but the method to solve it which has to do with geometry; and at times there is no evident affinity at all.

Some questions belong together in clusters, others are isolated. Not all of them are interesting for the present purpose.

One (isolated) question [B405;G622] – the very last – was already used above, because its use of *tetragonus* versus *quadratum* shows how Fibonacci deals with borrowed material.

Another isolated problem [B398;G611] is the classical two-tower problem: Between two towers with known distance is found a fountain, and two pigeons fly from the tops of the two towers at the same moment with the same speed, also arriving at the cup at the same time.



Fibonacci takes the heights of the two towers to be 30 and 50 paces, and their distance to be 50 paces. At first he gives a geometric solution, where *AB* and *GD* represents the towers. From the mid-point *E* of *AG* he constructs the orthogonal on the latter line. From the Pythagorean theorem its intersection *Z* with *BD* can be seen to be equidistant from *A* and *G*.

Afterwards, Fibonacci shows that *Z* will coincide with *B* if $AB^2 = BD^2+GD^2$, and outside *DB* if $AB^2 = BD^2+GD^2$. The appurtenant diagram is left-right oriented, as are the large majority of diagrams in the *Liber abbaci*, while the diagram supporting the basic argument is right-left oriented. This suggests that in this elaboration of the argument Fibonacci is on his own – the right-left orientation is almost certainly taken over from an Arabic source or from a Latin source reflecting its Arabic origin.

It *would* be possible to derive a numerical solution from the geometrical diagram, but it would be quite cumbersome if it were to reflect the geometric procedure directly. Therefore, “if you want to proceed by numbers”, Fibonacci presents an arithmetical rule.

The classical method, known since Mahāvīra’s *Gaṇita-Sāra-Sangraha* [ed., trans. Raṅgācārya 1912: 249*f*] and explained there, is as follows: $gd^2+dz^2 = gz^2 = az^2 = zb^2+ab^2$, whence $dz^2–zb^2 = ab^2–gd^2$, that is, $(dz–zb)\cdot(dz+zb) = (ab+gd)\cdot(ab–gd)$. But *ab*, *gd* and $dx+zb = bd$ are all known.

Fibonacci’s rule is

$$fz = \left(\frac{ab+gd}{2} \cdot \frac{ab - \frac{ab+gd}{2}}{2} \right) \div \frac{dz+zb}{2} \ .$$

which comes from $(DZ–ZB)·(DZ+ZB) = (ab+gd)·(ab–\frac{ab+gd}{2})$. Nothing in Fibonacci's text, however, suggests that he knows the basis for his rule, whereas there is no doubt that he understood the geometric argument.

The problem is also found in chapter 13, part 1, the part that contains problems that have already been dealt with (namely in chapter 12). More precisely, the two-tower problem is found in four manuscripts (**F**, **L** and two more belonging to the same family,[57] see [Giusti 2020: xlix]) with the same numerical parameters [B331;G519 app]. Since the problem is of the first degree, it can be solved by two false positions, $bz = 10$ and $bz = 15$.

The appearance in chapter 13 part 1 in spite from absence from chapter 12 persuaded Giusti to suppose the appearance in chapter 13 to be spurious. However, the problem is also found in chapter 12 *of* ***L***, which shows that the appearance in chapter 13 part 1 *of the 1202 edition* was justified. In **L** chapter 12 [ed. Giusti 2017: 192–194] the heights of the towers are 20 and 30, and their distance 26, that is, different from what we find in chapter 13 as well as chapter 15 of the 1228 version (and the pigeons are hawks, while their aim is a small bird). The rule is the same as the numerical rule in chapter 15 part 2 of the 1228 version, while the geometric argument is absent. It appears that the problem was present in chapter 12 of the 1202 edition, and there only solved by means of a rule whose origin Fibonacci did not know. When becoming aware of a geometric proof while revising the book, he inserted it with the new parameters found in his source, but also quite reasonably now moved the whole problem to the geometry part of chapter 15 – and then changed the parameters in chapter 13. Possibly, at first Fibonacci simply deleted the problem from chapter 13, since it was no longer dealt with in chapter 12, and then (the presence in one manuscript family only might point in that direction) reinserted it in chapter 13 with the new parameters in a second stage of the revision.

After the two-tower problem follows a cluster about repeated travels with constant profit rate (after expenses, or with no expenses) [B399;G612]. The first problem from the cluster states that somebody earned on 100 £

[57] **L** evidently only belongs to the family from chapter 13 onward.

at some market,[58] and then proportionally at the same rate when he travels to another one, ending up with 200 £; it is asked how much the merchant had after the first trip. This finding of the middle term in a continued proportion is illustrated by three lines carrying the letters *a*, *b* and *g*, and can be presumed to be borrowed faithfully (except presumably the final expression of √20000 £ in terms of £, ß and δ).

In the next problem, somebody enters in partnership after the first travel with 100 £, and after the second year the two together have 299 £; the question is the same. The solution to this mixed second-degree problem (formulated as a proportion) goes via the product rule and application of *Elements* II.6 in "key" version (neither "key" nor Euclid being mentioned), all supported by a line diagram lettered *a-b-g-d-c-e-z*. The most likely explanation is that Fibonacci was inspired by a borrowed problem but transformed the procedure ; the following problem, in any case, similar to the first one but this time with three travels, is without diagram but now solved with an unspecified reference to Euclid (actually *Elements* VIII.12, though here applied to irrational ratios); the extensive discourse to which this gives occasion is certainly of Fibonacci's own making. Whether the sequence was already present in 1202 or added in 1202 is difficult to say.

A rather peculiar problem type [B403–405;G618–620] treats of a cistern into which falls some object of given shape (cube, cylinder, double cone in distaff shape, sphere) and dimensions (thus giving its volume in cube feet); the contents of the cistern, on its part, is given in hollow measure (*bariles*). Similar problems are found in the *Liber mahameleth* [ed. Sesiano 2014: 536], showing rather unambiguously that Fibonacci drew here on an Iberian source.

Rules of algebra

The remaining problems in part 2 are not sufficiently characteristic to inform about our question (sometimes evidence is there but too ambiguous); I shall therefore leave them aside and turn to the first section of chapter 15 part 3.

The heading of part 3 [B406;G622] states it to deal with *solutione quarumdam questionum secundum modum algebre et almuchabale, scilicet appositionis et restaurationis*, "the solution of certain questions according

[58] I use £ for the monetary unit *libra*, ß for *soldus* and δ for *denarius*, 1 £ = 20 ß, 1 ß = 12 δ. Readers will recognize the system that remained in use in Britain until 1971.

to the way of *algebra et almuchabala*, that is, of apposition[59] and restoration". As we see, this Latin explanation of *algebra et almuchabala* differs from the one given in the preamble to chapter 14, showing that now a different source is drawn upon. The inversion – it is *algebra* that translates *restoration*, while *almuchabala* should become "apposition" or "opposition" – shows that Fibonacci does not translate directly himself from an Arabic source; whether he keeps the order of his source because of insufficient Arabic proficiency or rather as another instance of faithfulness to his source is an open question.

What comes after the heading – [B406;G622] onward – corresponds to one component of the initial part of al-Khwārizmī's algebra, *Al-kitāb al-mukhtasar fī hisāb al-jabr wa'l-muqābalah*;[60] what al-Khwārizmī teaches about the determination of square roots has already been covered in greater depth in chapter 14 of the *Liber abbaci*, and Fibonacci may have thought that what he has explained there about the arithmetic of arithmetical binomials can be transferred directly to that of algebraic binomials.

Even that which corresponds to al-Khwārizmī's introduction – namely the rules for solving the six basic equations and the geometric demonstrations – is not copied from this classic. Admittedly, Nobuo Miura [1981] has pointed to a number of characteristic phrases that show familiarity with Gerard of Cremona's translation of that work; but this rather implies that Fibonacci works on his own on the basis of his source, if not there could be no traces of Gerard's text. This is confirmed by the *a-b-c* lettering of most of the geometric diagrams used to prove the rules.

Of particular interest is the first geometric demonstration for the case

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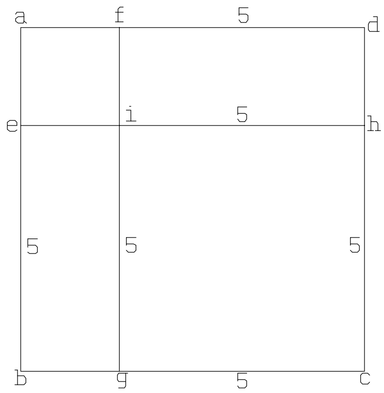
59 *Appositionis*, probably a miswriting for *oppositionis* but possibly an alternative translation that would mean "setting before". Boncompagni, following his manuscript, has *ad proportionem*, obviously an attempt to repair the impossible grammar of two other manuscripts having "a proportionis" [Giusti 2020: 808]. Two attempts to explain Boncompagni's *ad proportionem* ([Hughes 2004: 324 n. 43] and [Høyrup 2011: 94*f*] can now be happily discarded.

60 Alternatively, as argued by Roshdi Rashed [2007: 9], *Kitāb al-jabr wa'l-muqābalah*. My references for the comparison will be to Gerard of Cremona's Latin translation [ed. Hughes 1986] because it is closer to the Arabic original than the extant Arabic texts, all later by a small century or more [Høyrup 1998; Rashed 2007: 83, 86]. When referring to the Arabic text I shall use [Rashed 2007].

“*census* and given roots made equal to a given number” [B407*f*;G624],[61] exemplified by “a *census* and 10 *roots* are made equal to 39” (like al-Khwārizmī, Fibonacci offers two demonstrations, though different). Beyond the lettering the proofs are introduced in a way that suggests Fibonacci’s own work,

> *unde hec regula procedat, per duplicem figuram ostendere procurabo*
>
> from where this rule comes, I shall undertake to show by a double diagram.



The diagram is similar to al-Khwārizmī’s second demonstration, but there is a fundamental difference of mathematical style. Al-Khwārizmī starts “from the inside” with the square *ihcg* (obviously using other letters), whose sides are stated to be 5. Fibonacci starts “from the outside”, with *abcd*, whose sides are claimed to be *amplius quam ulnas 5*, “longer than 5 cubits”. So, whereas al-Khwārizmī’s demonstration is analytic, that of Fibonacci in synthetic; and its inspiration is *Elements* II.4. It is highly implausible that Fibonacci should have presented algebra without geometric proofs in 1202 (at least from Gerard’s translation he knew about the possibility); we may conclude that he had a Euclidean formation already by then (should anybody have doubted it – as we remember, Euclid is also spoken of in what appears to be the 1202 prologue to the whole work).

The second demonstration builds on *Elements* II.6, and *could* be a reduced version of the one given by Abū Kāmil [ed. trans. Rashed 2012: 254]. Abū Kāmil’s algebra was known in 12th-century al-Andalus, as can be seen from correct references to that text in the *Liber mahameleth;* an indirect inspiration is more likely, however (as we shall see).

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[61] Some fundamentals for those who are not familiar with medieval Arabic and Latin algebra: its core is a second-degree technique formulated around a *māl* (in Latin *census*), formally a monetary possession, and its square root; accordingly, the number term is spoken of as a number of *dirham* / *dragma*. Already al-Khwārizmī, who wrote the earliest treatise about the topic (indicating in his introduction that he describes a technique that was already around), treats the *root* as the basic unknown, even though he also treats the *māl* / *census* as an unknown that has to be found.

99 algebra questions

The larger portion of chapter 15 part 3 is constituted by a collection of 99 problems, with occasional theoretical insertions.[62] Many of them share the mathematical structure of problems found in al-Khwārizmī, regularly but far from always also the numerical parameters. Many (sometimes the same, sometimes others) coincide in structure with problems from Abū Kāmil's *Algebra* in one or both ways.

The many cases where structures are shared but numerical parameters are different suggest – not least in view of Fibonacci's tendency to be faithful to his sources – that the borrowings are indirect. The rather few partial agreements with al-Karajī's *Fakhrī* (structure of problem, but rarely parameters or procedure) cannot be taken as evidence that Fibonacci knew that work.

A strong suggestion that borrowings from Abū Kāmil are indirect is offered by the problem [H#21;G§288], one of 32 problems about a "divided 10". In letter formalism:

$$10 = a+b \ , \ \ ({}^{a}/_{b}+10)\cdot({}^{b}/_{a}+10) = 122\,{}^{2}/_{3}$$

Abū Kāmil [ed. trans. Rashed 2012: 410*f*] solves the same problem; al-Karajī instead gives the sum as $143\,{}^{1}/_{2}$ [Woepcke 1852: 94]. As far as can be read out of Franz Woepcke's paraphrase, al-Karajī posits that a is a *thing;* a simple transformation then reduces the problem to "*census* plus 16 made equal to 10 roots", on of the standard cases. Abū Kāmil posits ${}^{a}/_{b}$ to be a "large thing" (presupposing $a>b$), and ${}^{b}/_{a}$ to be a "small thing". Then, if R stands for the "large thing" and r for the "small thing",

$$(R+10)\cdot(r+10) = 122\,{}^{2}/_{3} \ ,$$

and since $rR = 1$,

$$1+10\cdot(R+r)+100 = 122\,{}^{2}/_{3} \ ,$$

whence

---

62 The precise number depends on which variants are counted as independent questions. I follow the list in [Hughes 2004: 350–361], which along with [Boncompagni 1857] draws on the edition of chapter 15 in [Libri 1838: II, 307–479], based on a different manuscript of the *Liber abbaci*, and on Benedetto da Firenze's vernacular translation of the questions as rendered in [Salomone 1984]. A problem referred to as [H#*m*;G§*n*] is number *m* in Barnabas Hughes' list and §XV.*n* in [Giusti 2020], [G§*n*] refers to §XV.*n* in [Giusti 2020].

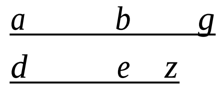
$$R+r = 2^1/_6 \ .$$

That is, the problem is reduced to

$$10 = a+b \ , \ \ ^a/_b+^b/_a = 2^1/_6 \ ,$$

which Abū Kāmil has already dealt with.

Fibonacci uses a line diagram, lettered *a-b-g-d-e-z*. Here, $ab = de = 10$, while $bg = {}^a/_b$, $ez = {}^b/_a$. Abū Kāmil's two algebraic unknowns are thus replaced by line segments.



The procedure is parallel to that of Abū Kāmil, and also leads to the same reference to what has already been dealt with – actually a reference to what has been dealt with by Fibonacci's source! Fibonacci himself [H#10,243; G§243] has treated the case where the sum of the two fractions is $3^1/_3$, not $2^1/_6$ (cf. below, note 64). A clear indication of copying – not directly from Abū Kāmil, however, but at most (and, in view of the shared structure of the argument, probably) from a source building on but reshaping Abū Kāmil's solution.

In the end Fibonacci says that the reader should know that

> when you have two numbers and divide the larger by the smaller and the smaller by the larger and multiply that which resulted from one division in that which resulted from the other, then from their multiplication always 1 is generated, and therefore I said 1 to come from *bg* in *ez*.

As shown by the *a-b-g*-lettering, we have a parallel to what we saw in the *tetragonus*/*quadratus* problem [B405;G622]: a faithfully borrowed text, supplemented by a personal explanation given afterwards, not integrated in what was taken over.

Al-Karajī, as we notice, offers a typical *al-jabr* solution. Abū Kāmil's reduction makes use of a technique rather belonging with the *regula recta* with two unknowns (the problem to which he reduces the present one is then solved by means of *al-jabr*); Fibonacci, and his source, also remove anything that could make one think of *al-jabr* techniques (with the same proviso).

If one looks at the problem collection as a whole it becomes obvious that it does not guide the reader systematically from simple to more advanced or difficult matters. Instead groups of problems have been adopted together from the same source. In some cases, this source can be identified with some approximation, in others even that is impossible.

The first eleven problems exemplify the first type. Nine of them have a counterpart in the beginning of al-Khwārizmī's algebra, that is, they have the same mathematical structure: five in his list of six illustrations of the basic cases, four in his collection of "various problems". Internally in each of these groups, they follow al-Khwārizmī's order, but the two groups are mixed up.[63] Of the nine that have a counterpart, that is, the same mathematical structure, only two share al-Khwārizmī's numerical parameters; only one [H#10;G§243] has the same initial formulation as Gerard of Cremona's translation of al-Khwārizmī, which however is so simple that the coincidence might well be accidental; but in that case the numerical parameters are different, and the procedure is quite different.[64] This excludes that Fibonacci should have used al-Khwārizmī's *Algebra* (in Gerard's or any other version) directly for this sequence; on the other hand it shows that he drew on an introductory work descending from that model (produced by a writer who was less faithful to his sources). The similarity of the demonstration of [H#10;G§243] to that of [H#21;G§288] suggests that the two borrowed from writings belonging to the same school of thought; in view of the stylistically different diagrams hardly from a single treatise, however.

Another cluster of problems is characterized by the appearance of an *avere*, a Romance (Italian, Catalan, Provençal or Castilian) loanword meaning "possession". It obviously translates *māl* (above, notes 31 and 61) – but only when this term is used about an unknown quantity, literally an amount of money. Its first appearance is in [H#62;G§387]:

Further, I multiplied the root of the sextuple of some *avere* in the root of

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63 With Q referring to the six illustrating questions, V to the *varia*, and – indicating absence of a counterpart, Fibonacci's order is V1, –, Q2, Q3, –, Q4, Q5, V2, Q6, V4, V5. Using a simple combinatorial model we find that the odds that the order of borrowings from the two groups should be conserved by accident is $^1/_{4!\cdot 5!}$ = $^1/_{2880}$

64 The problem is

$$10 = a+b\ ,\quad {}^a/_b+{}^b/_a = 3\,^1/_3\ ,$$

almost the same as the one to which the problem

$$10 = a+b\ ,\quad ({}^a/_b+10)\cdot({}^b/_a+10) = 122\,^2/_3$$

[H#21;G§288] was reduced, just with the sum being $2\,^1/_6$ (which is also the sum in al-Khwārizmī's version of the present problem). That problem (coming later in the *Liber abbaci*) is reduced, we remember, by means of line segments representing *a*, *b*, $^a/_b$ and $^b/_a$, respectively. A similar strategy is used here, though with the four segments being separate and each designated by a single letter (*a*, *b*, *g* and *d*). Al-Khwārizmī, who never uses line representations in the problem solutions, evidently has none here.

its quintuple, and I added the decuple of the same *avere* and 20 *denarii*, and all this was as the multiplication of the same *avere* in itself. I shall posit for the same a *thing*, and I shall multiply the root of its sextuple in the root of its quintuple, that is, the root of 6 *things* in the root of 5 *things*. The root of 20 *census* results, since when a *thing* is multiplied in a *thing* it makes a *census*, whence when the root of a *thing* is multiplied in the root of a *thing* the root of a *census* results. Then I shall add above the root of 30 *census* the decuple of a *thing* and 20 *denarii*, and I shall have 10 *things* and the root of 30 *census* and 20 *denarii*, which is made equal to the multiplication of a *thing* in itself, that is, a *census*. In this falls the rule of roots and numbers which are made equal to a *census*.

The concluding statement presupposes that $\surd(30C)+10r$ is understood to be $(10+\surd30)r$. This, however, would not be acceptable according to the canon that only integers and, in practice, rational fractions were accepted as numbers and hence as coefficients.[65]

We shall return to this, and to the way the problem is circumvented, but at first look at the *avere* cluster as a whole.

Avere reappears in 13 further problems.[66] Sometimes this initial *avere* is posited afterwards to be a *thing*, sometimes to be a *census*. That obviously depends on what will yield a convenient equation, and does not tell us more than that.

More interesting is that all of these constitute a closed group, adopted from the same source. The apparent interruptions in the sequence all deal, either with a divided 10 (once a divided 12) or with *two* numbers or quantities; therefore they would not allow the appearance of any substitute for *māl* in this sense – *avere* or otherwise. Nor would any of the problems that follow except the very last [H#99;G§682];[67] the cluster may therefore

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[65] This canon was respected from al-Khwārizmī until the European 16th century, see [Oaks 2017]. Since the difficulty was seen to be an obstacle that was to be, and – as we see here – *was* circumvented, the avoidance of irrationals as coefficients *was* a canon, and not the result of failing understanding of possibilities; cf. [Høyrup 2004].

[66] [H#66;G§410], [B70;G439], [H#76;G§531], [H#77;G§539], [H#78;G§543], [H#79;G§546], [H#80;G§549], [H#81;G§551], [H#82;G§554], [H#83;G§557], [H#84;G§559], [H#85;G§561], [H#87;G§570].

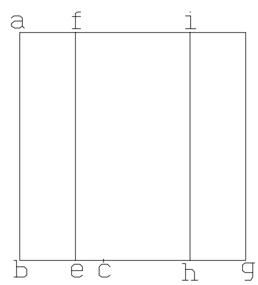
[67] A simple first-degree problem, "I multiplied the 30-double of a *census* by 30 and what resulted was equal to the addition of 30 dragmas and the 30-double of the same *census*" – noteworthy at most (but hardly) for the use of *additio* in the sense of sum, which is unique in the algebra section though found in the last problem

well have extended further (as we shall see, there is more evidence for that).

This source must already have used the term *avere*. There is no reason that Fibonacci should suddenly on his own choose a new translation, and then give it up in the last problem – earlier problems use the standard translation *census* for *māl* in both roles; nor any reason that elsewhere but not here he should replace an original Arabic initial *māl* by *numerus* or *quantitas*. We cannot exclude that this source was already written in a Romance vernacular (Italian, Catalan, Provençal or Castilian, though Italian seems even more unlikely than the others); more plausible is a Latin translation prepared in a Romance-speaking environment which borrows terms from the local vernacular (still Catalan, Provençal or Castilian, hardly any Italian vernacular).[68] Since we already encountered one source corresponding to this which also makes use of a non-standard terminology, namely in the preamble to chapter 14, we might suppose it to be the same – but only if we find it quite improbable that not one but *two* such treatises should have disappeared, or at least disappeared from view (my formulation should betray that I do not find it improbable).

Other clusters can be suspected, and below we shall look at one of them. First, however, we shall return to [H#62;G§387] and look at how Fibonacci manages to circumvent the difficulty that he is not allowed to apply the rule he has seen to be pertinent because he has encountered an irrational coefficient. His text goes on:

> In order to show that, let there be placed hereby an equilateral and equiangular quadrangle *ag*, whose side is *bg*, and posit *bg* to be a *thing*. Therefore we cut off from the square *ag* a rectangular surface *ae*, which should be root of 30 *census*, and from the surface *fg* is removed the surface *fh*, which should be equal to 10 roots of the *census ag*, wherefore *eh* is 10. From the whole square *ag* remains the surface *ig*, which will be 20. And because the surface *ae* is the root of 30 *census* and comes from the multiplication of *ab* in *be*, and *ab* is a *thing*, it follows by necessity that *be* must be the root of 30, since from the multiplication of a *thing*



of chapter 15 section 2 and occasionally in chapter 14. It seems to be a trace of *something* but of exactly *what* we cannot guess.

68 Fibonacci too adopts vernacular terms regularly – see below, note 81, for an example.

in the root of 30 results 30 *census*. We add thus *be* with *eh*, and the whole *bh* will be 10 and root of 30, which is a fourth binomial; and we divide it in two equals at he point *c*, and each of the lines *bc* and *ch* will be 5 and the root of $7^1/_2$. And because the surface *ig* is 20, that which results from the multiplication of *ih* in *hg*, that is, from *bg* in *hg*, if above 20 we add the multiplication from *ch* in itself, which is $32^1/_2$ and the root of 750, we shall have $52^1/_2$ and the root of 750 for the square on the line *cg*. Then *cg* is the root of $52^1/_2$ and the root of 750, and if we add to it the line *cb* we shall have for the whole *bg*, that is, for the requested *avere*, the root of $52^1/_2$ and the root of 750 and 5δ and the root of $7^1/_2$δ; all of which is according to approximation around $16^2/_3$.

The argument makes use of *Elements* II.6 or of the corresponding "key" (none of which are mentioned), according to which $bh{\cdot}hg+bc{\cdot}bc = cg{\cdot}cg$. $bh{\cdot}hg$ indeed equals $ih{\cdot}hg$ and therefore 20. As we see, the rule for the case roots and number made equal to *census* is spoken about no longer, and the (correct but redundant) observation that $10+\sqrt{30}$ is a fourth binomial points back to the secondary layer of chapter 14. As we see, the difficulty of irrational coefficients is eschewed by the application of geometry.[69]

The lettering, including a *c* entering late in the argument, suggests that Fibonacci has intervened himself. This is confirmed by a comparison with the preceding problem [H#61;G§383], $\sqrt{(8n)}\cdot\sqrt{(3n)}+20 = n^2$ (*n* being there a *numerus*, no *avere*). This *n* is directly identified with a line *bg* and n^2 with the corresponding square, here spoken of as a *tetragonus*. In that case, the lettering is *b–g–d–f–h* – *a* being left out because the corresponding corner of the square is not mentioned. There a binomial (in that case the result) is identified as a *sixth binomial*, whereas the problem before that [H#60; G§381], $(8\sqrt{n})\cdot(3\sqrt{n})+20= n^2$ has a *fifth binomial* (and no diagram, since the problem reduces to $24n+20 = n^2$, with no irrational coefficients occurring). No other references to the classes of *Elements* X occur in the collection of algebraic questions. It appears that in [H#62;G§387] Fibonacci has borrowed a proof technique from [H#61;G§383] that does not belong to the *avere* cluster, and then adapted the proof to the situation where the coefficient is $10+\sqrt{30}$ and no simple root, employing also his own terminology ("an equilateral and equiangular quadrangle" then becoming simply "square", *quadratus*). As we have seen it before, Fibonacci shifts to his own language

69 The final approximation is worth observing. Approximation is used nowhere else in Fibonacci's algebra, but obviously dealt with extensively in the primary layer of chapter 14.

when he is creative, and avoids imitation.

The following problem ([H#63;G§392], still belonging to the *avere*-cluster) and dealing with a "divided 10", also leads to an irrational coefficient; a diagram lettered *a-b-c-d-e-g* is made use of. Even here, and perhaps more radically, Fibonacci seems to work independently. Once more, the square is spoken of as *quadratum equilaterum et equiangulum*.[70] Most other diagrams from the cluster are also of *a-b-c*-type.

A last cluster to inform us about Fibonacci's working habits contains problems about an amount of money first divided among a number of men, then either the same or a different amount divided among more men. The first of them [H#12;G§252] runs:

> I divided 60 between some men, and something resulted for each; and I added two men above them, and between all these I divided 60, and for each resulted $2^1/_2$ less than resulted at first. Let the number of the first men be the line *ab*, and on it is erected at a right angle the line *bg*, which should be that which falls to each of them of the mentioned 60δ, and draw the line *gd* equal and parallel to the line *ba*, and the straight line *da* is connected. Then the space of the quadrangle *abgd* will be 60, when it is brought together [*colligatur*] by *ab* in *gb*. Then protract the line *ab* to the point *e*, and let *be* be 2, that is, the number of men to be added. And on the line *bg* the point *f* is marked, and let *gf* be $2^1/_2$, that is, that which each one got less by the addition of two men. And through the point *f* the line *hi* is protracted equal and parallel to the line *ea*, and the straight line *eh* is connected; the quadrangle *heai* will be 60, since it is contained by *ae* in *eh*, namely by *ae* in *bf*, where *bf* is that which resulted for each of the men *ae* from the 60δ. The surface *ei* is thus made equal to the surface *bd*. The multiplication of *gb* in *ba* is thus made equal to the multiplication of *ea* in *fb*. Whence these four lines are proportional. Therefore, the first *gb* is to the second *fb* as the third *ea* to the fourth *ba*, whence, by dividing,[71] as *gf* is to *fb*, so is *fb* to *ba*. But the ratio *gf* to *eb* is as 5 to 4. Thus *fb* contains once and one fourth the number *ba*.
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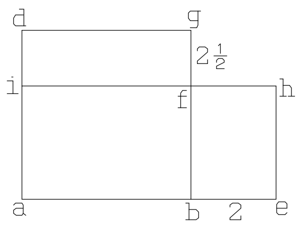
> 
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>
> So, posit for the number *ab* a *thing*. *bf* will thus be $1^1/_4$ *thing;* and

70 However, the use of *tetragonus* versus "equilateral and equiangular quadrangle" is not quite systematically coupled to diagrams lettered *a-b-g-...* respectively *a-b-c-...* .

71 That is, we transform $\frac{gb}{fb} = \frac{ea}{ba}$ into $\frac{gb-fb}{fb} = \frac{ea-ba}{ba}$, from which follows that $\frac{gf}{fb} = \frac{eb}{ba}$. That could, by the way, be seen directly in the diagram, just by removal of the shared surface *af* from both of the surfaces *ag* and *ah*. "Permutation" leads to $\frac{gf}{en} = \frac{fb}{ba}$.

multiply *ab* in *bf*, and $1^1/_4$ *census* results for the surface *bi* [...].

Al-Khwārizmī's own version of his algebra contains no problems of this kind – neither as reflected in Gerard of Cremona's translation, nor in Robert of Chester's somewhat extended version [ed. Hughes 1989]. In the Arabic manuscripts (all later than the two translations[72]) we have a version where the amount to be distributed is 1 dirham, only one man is added, and the difference is $^1/_6$ [ed. trans. Rashed 2007: 190*f*]. Even there, the solution consists of several parts. First there is a description in general terms; this description appears to correspond to a diagram which however has disappeared. Then the same question but formulated as a paradigmatic example with explicit numerical values. Finally, as in Fibonacci's problem, the solution of the resulting equation. However, this may have crept into the tradition at any moment before 1222, and there is no reason to believe it inspired Fibonacci, neither directly nor indirectly.

What Fibonacci proposes is instead preceded by a similar problem in Abū Kāmil's algebra [ed. trans. Rashed 2012: 352–355]. Here, 50 dirham are shared first among some men, them among 3 more, the difference between what each one gets in the two situations being $3^3/_4$ dirhams. The solution follows the same pattern as that of Fibonacci, but instead of using proportions the argument about the diagram is arithmetical all the way through; the diagram that appears to have been lost from the Arabic "al-Khwārizmī" algebra could be similar.

Next in the *Liber abbaci* follows a problem [H#13;G§259] where first 20 is divided between some number of men, next 30 between 3 more, the difference between the shares in the two situations being 4. The solution is based on a diagram of the same character though slightly more complicated, lettered *a-b-g-d-e-...*, and on proportion techniques (followed by algebraic solution of the resulting equation). Once again, Abū Kāmil offers four problem of the same structure [ed. trans. Rashed 2012: 358–371], presenting solutions based on diagrams that are similar to that of the *Liber abbaci;* proportions are not mentioned.

The following two problems in the *Liber abbaci* [H#14–15;G§266,271] have the same mathematical structure. This time, however, a diagram lettered *a-b-c-d-e-...* is used, and the algebraic entities (*thing* and *census*) enter directly in the discussion of the diagram, and proportions are not

[72] The oldest is dated 1222 [Rashed 2007: 85].

referred to. To judge from the different lettering of the diagrams in these last three problems, the first builds on a source that ultimately goes back to Abū Kāmil, while the other two are of Fibonacci's own brew. The first reformulates Abū Kāmil's solution in proportion terms; Fibonacci, in his own (more straightforward) solution, does not mention them.

In the last problem from the sequence [H#16;G§276], 10 are divided between a certain number of men and then 40 between 6 more; they get the same in the two cases. Anybody tending to think in terms of proportions would state this as $\frac{h+6}{h} = \frac{40}{10}$, from which would follow $\frac{6}{h} = \frac{40-10}{10}$, thus $6\cdot10 = 30\cdot h$. But Fibonacci, again seemingly working on his own[73] has no such preferences on the present occasion. He just observes (thus not using algebra) that the 30 extra monetary units must be the share of the 6 extra men, each of whom therefore gets 5. Since the first men get the same, their number must be $10\div5 = 2$.

There can be no doubt that the sequence [H#12–16] is part of a cluster adopted from a single source; for the last three problems, however, Fibonacci seems to have presented a simpler solution of his own making. Since [H#11] belongs to the cluster borrowed indirectly from al-Khwārizmī's algebra, [H#12] is the first member of the present cluster; whether it extends beyond [H#16] seems undecidable (but rather unlikely according to internal criteria of style). The propensity of the source for the cluster to reformulate Abū Kāmil's geometry in proportion terms may make us think of other borrowings pointing to al-Andalus.

A final question to address in this discussion of Fibonacci's algebra is whether and how he deals with higher-degree problems. This topic has often been misrepresented, and can be somewhat clarified on the basis of what has been said.

First it has to be pointed out that many of the problems that have been considered biquadratic by earlier workers only become so because of failure to understand the distinction between the two roles of *census* – for instance [H#44;G§344]:

[73] The problem is also in Abū Kāmil's algebra [ed. trans. Rashed 2012: 370–373]. First Abū Kāmil gives two unexplained numerical prescriptions, the second of which corresponds to Fibonacci's solution; about the latter it is said that "the reason of that is obvious"; next he formulates the proportion $\frac{h}{h+6} = \frac{10}{40}$, and then identifies the second ratio with the number $^1/_4$. Positing *h* to be a *thing* he gets an algebraic equation.

I multiplied the third of a census and $1^3/_4$ in its fourth and $2^3/_4$, and a census augmented by $13^3/_4$ resulted. Posit a thing for the census. [...].

If it is not realized that the initial *census* is of the kind that elsewhere is sometimes spoken of as an *avere* or a *number* and really means an "amount of money", this looks like a biquadratic solved by means of a substitution of variable. When no positing is needed – for instance, in [H#38;G§340] – the *census* in question is indeed considered the solution, its root is not found, which confirms its meaning as an "amount". There are more of these, and there is no reason to discuss them any further.

Others, however, are properly biquadratic or bi-biquadratic, or lead to solvable third-degree equations. *They are all found within the "extended avere cluster"*, confirming the suspicion that this (including the extensions) really *is* a group.[74] Here, a corresponding terminology is also used (*cubus*, *cubus cubi*, *census census*, *census census census*, *census census census census*, for the third, the sixth, the fourth, the sixth and the eighth power); and scattered theoretical observations though no systematic presentation of higher-degree techniques can be found.

We shall look at a single example [H#88;G§575], in which Fibonacci appears to have intervened actively with a justification:

Of three unequal quantities, when the major and the minor are multiplied it is as the middle in itself, and when the major is multiplied in itself results as much as the minor in itself and the middle in itself joined,[75] and from the multiplication of the minor in the middle results 10. Posit for the smaller a *thing* and for the middle 10 divided by a *thing*, and multiply 10 divided by a *thing* by itself, and 100 divided by a *census* results, which you divide by a *thing:* 100 divided by a *cube* result, and this will be the major quantity. Then multiply the minor quantity, namely a *thing*, in itself, and a *census* results; and multiply the middle in itself, namely 10 divided by a *thing*. 100 divided by a *census* results, which you shall add with the *census*, they will be a *census* and 100 divided by a *census*, which is made equal to the multiplication of the major quantity, namely 100 divided by a *cube* in itself, from which multiplication result 10000 divided by a *cube of cube*. Then multiply everything you have by *cube of cube;* and to multiply by *cube of*

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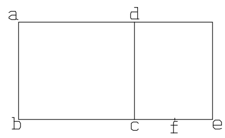
[74] Since this cluster is found close to the end of the chapter, we may get an impression of theoretical progress. This impression, however, is an artefact, and the very final trivial first-degree problem disproves it – sese above, note 67.

[75] In other words, the three numbers are in continued proportion, and may be sides of a right triangle.

> *cube* is as multiplying by *census of census of census*. Then if we multiply 10000 divided by *cube of cube* by *census of census of census*, 10000 result; and if we multiply a *census*, namely the square of the minor quantity, by *census of census of census*, we shall therefore have a *census of census of census of census;* and if we multiply the square of the middle quantity, namely 100 divided by *census*, by *census of census of census*, results *100 census of census*. Therefore a *census of census of census of census* and 100 *census of census* are made equal to 10000 dragmas.

At this point, we might perhaps have expected Fibonacci to posit a (new) *thing* for the *census of census*, or simply to have applied the standard rule for the case "*census* and *things* made equal to number". Instead, as when application of the standard algorithms would presuppose the explicit use of irrational coefficients in [H#62;G§387] and [H#63;G§382], Fibonacci uses the geometric configuration that serves to justify the rule in question, positing the square *ac* for the *census of census of census of census*. The lettering as well as the use of *quadratus* indicates that the proof was inserted by Fibonacci himself (the two proofs circumventing irrational coefficients, as we remember, were also characterized by the lettering *a-b-c-...*, and also used *quadratus*). Together, Fibonacci's need to intervene actively in these three cases[76] suggests that his source for the *avere* group had fewer qualms with irrational coefficients than he had himself and handled higher powers more freely.



In both cases, Fibonacci's independent construction of proofs shows that he had no difficulty in understanding what his source was doing. That seems to hold throughout the algebra-part, with a single exception: an alternative solution to [H#71;G§448], where he does not understand the Arabic habit of using coin names as second, third and fourth algebraic unknowns. Since I have discussed that problem in detail in [Høyrup 2019: 32–35], I shall not do it here[77] – only add that the alternative solution, and thus the whole problem, is obviously a borrowing. However, since

[76] Supported by [H#63;G§392], [H#89;G§583], [H#71;G§448] and [H#73;G§497], which share the same characteristics. The last of them has the letter sequence *c–d–e–f–g*, *a* and *b* having been used already as one-letter line-carried symbols for $^{10-r}/_{r}$ and $^{r}/_{10-r}$, respectively, similarly to Abū Kāmil's "large" and "small thing".

[77] Presupposing Fibonacci to have intended the right solution, Giusti [2020: 660] corrects the text. His apparatus (as well as [Boncompagni 1857: 435]) shows what is really in the text.

the problem belongs to the “extended *avere* cluster”, this is nothing new.

In summary, Fibonacci’s algebra as presented in chapter 15 part 3 is no *treatise*, no new coherent and systematic presentation of the field. It is an anthology, a collection of excerpts from other texts (with an introduction that leaves out what has already been dealt with in chapter 14, and with interspersed additional explanations). With a single exception, everything is well understood by Fibonacci. It was neither an elementary introduction nor a systematically progressing guide to advanced methods. It fits exactly what was promised in the prologue.

Particular problems from chapter 12

From this survey of whole chapters we shall now turn to select single problems from chapter 12.

First [B298;G471]:

> Somebody gave somebody for his daily work 1 mark of silver, which he paid by means of five cups that he had, so that none of them was broken; and this he did for 30 days. The weight of the first cup was 1, whose double, namely 2 mark, was the weight of the second. The weight of the third was 4, namely the double of the second. But the weight of the fourth was the double of the third, namely 8. When the weights of these 4 cups are joined together, they make 15 mark. When these are extracted from 30 mark, 15 mark remain for the weight of the fifth cup. On the first day he gave him the first vase. On the second he received from him this same first, and gave him the second. On the third the lord gave the worker this same first. On the fourth the lord received from the worker the first and the second, and gave him the third. And thus in the said order he paid him daily, until 30 days.

This builds on the insight that any integer can be expressed unequivocally as a sum of powers of 2 (the final 15 instead of 16 being chosen as a pragmatic shortcut allowing the worker to leave with all the cups). That is old knowledge, it underlies the Pharaonic standard multiplication algorithm.[78] Interesting is instead the term used for the cup. In **F** it is *sisphos* and *ciphus*, in **L** [ed. Giusti 2017: 184] it is *sciphos* and *scifis*. This is neither Latin nor borrowed from any Romance language. It renders *spoken* Byzantine Greek, namely the way σκύφος was pronounced (better

[78] It follows from the observation that division by 2 leaves either remainder 0 or remainder 1.

in **L** than in **F**, where the spelling is further influenced by Tuscan pronunciation). That is, Fibonacci encountered the problem in oral interaction in Byzantium.[79]

This problem is preceded [B297;G471] by what is generally known as "Bachet's weight problem" (on whose history see [Knobloch 1973]). The earliest known appearance of the problem is Mohammad ibn Ayyūb al-Ṭabarī's *Miftāḥ al-mu ʿāmalāt* from c. 1100:[80]

> Somebody had 4 weight pieces [*pesones*[81]], by which he weighed whole pounds of his merchandise from one pound until 40 pounds. The weight [*pondus*] of each of these weight pieces is asked for. Then it is necessary that the first be of one pound; so that by it one pound can be measured. The second must be its double, with one added, or the triple of the same first; with these two weight pieces can be weighed from one pound until 4. But the weight of the third is one more than the double of both the others, that is, the triple of the second, namely 9; but the weight of the fourth is 1 more than the weight of the other three, that is, the triple of the third, namely 27; the weights of which joined together make 40. So, if you want to know how you may weigh with these weight pieces any number of pounds from one pound to 40 pound, let us say 14, then the fourth weight piece is put into one scale pan, and the rest is put in the other; and if you put the same fourth weight piece together with the first, and if you put in the other the rest, namely 9 and 3, then 16 pounds may be weighed [...]. And if you add a fifth weight piece, whose weight is the triple of that of the fourth, namely 81, with these five weight pieces may be weighed any number of pounds from one pound until 121 pounds; and thus in the same order weight pieces may be added without end.

Since Fibonacci groups this together with the cup-problem, he probably saw them as related – from our point of view they are, both being connected to partition theory. It seems to be a natural assumption (it can be no more) that *both* came from the same source, and thus from Byzantium. *If* we accept this, however, other plausible consequences follows. The final clause of the weight problem, *sic eodem ordine possunt*

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79 This agrees with the presence of a closely related problem in an early 14th-century Byzantine problem collection [ed. trans. Vogel 1968: 112*f*], which seems not to be influenced by Italian material.

80 [Tropfke / Vogel et al 1980: 634], date according to [Hockey, Trimble & Williams 2007: 1149].

81 One of many cases where Fibonacci borrows and Latinizes a vernacular term (here Tuscan *peso*), as commonly done in Medieval Latin.

addi pesones ad infinitum, is very close to that of the famous rabbit problem [B283;G453], *et sic posses facere per ordinem de infinitis mensibus*, "and thus you can do in order for infinite months". That problem, on its part, follows immediately after the rule for the production of perfect numbers [B283; G452], coming evidently from *Elements* IX.36 but expressed in the Byzantine terminology of the time, prime numbers being *sine regula*, "without rule",[82] although Fibonacci shows elsewhere to master the Euclidean terminology to perfection. Even here we may presume Fibonacci to be faithful to his source. All three problems and the rule thus seem to have come from (likely oral) interaction in Byzantium.

Unconnected to these is the "chessboard problem", [B486;G309] onward. Fibonacci explains that

> The doubling of the chess-board is proposed in two ways, of which one is that the following square [*punctum*] is double its antecedent; the other, when the following square is the double of all its antecedents.

Both possibilities are explored with details and perspective. The first begins in this way:

> The first doubling can be made in two ways, namely if we operate by doubling from square to square until the last square. The other way is that you double as much as the first square, and you have two; which two multiply in itself, they will be 4; which 4 are 1 more than the number of the doublings[83] of the two squares. For example: In the first square put 1. In the second 2; which joined, make 3; the above-written 4 are 1 more than these three; when these 4 are multiplied in themselves, they make 16; which number is one more than the doublings of the double of the first two doublings, that is, of 4 squares. For example: In the first there is 1. In the second 2. In the third, 4. In the fourth 8; which, joined together, make 15; which is 1 less than 16. Further multiply 16 in itself, they make 256; which are 1 more than the number of doublings of the double of the above-written squares, that is, of 8 squares which occupy the first row of the chess-board. For example, in the first there is one. In the second 2. In the

[82] That this is (a loan translation of) contemporary Byzantine terminology is explained by Fibonacci himself [B30;G53]: *Arabes ipsos* [*numeri incompositi*] *hasam appellant, Greci coris canonos, nos autem sine regulas eos appellamus*, "the Arabs call them [noncomposite numbers] *hasam*, the Greek *coris canonos*, we however call them 'without rule'" – *regula* / "rule" meaning factorization.

[83] From square 2 onward, the contents of a square legitimately can be spoken of as a "doubling" (*duplicatio*); Fibonacci extends the usage to the first square.

> third 4. In the fourth 8. In the fifth 16. In the sixth 32. In the seventh 64. In the eighth 128; which joined together make 255; which the above-written 256 exceed by 1, as we have said: therefore multiply 256 in itself, they make 65536, one more than the doublings of the first two rows, namely of 16 squares.

Fibonacci then finds “one more than the doublings” of the first four rows, then of all eight lines of the chess-board, and then of two chess-boards. “And multiplying thus we can go on until infinity”.

Nothing sensational – all that is done here depends on well-known properties of continued proportions (not referred to here, it is true).

Next Fibonacci offers a pedagogical illustration because, as he says, the resulting huge numbers may be difficult to grasp. He suggests to fill a *chest* with the contents of the first two rows (augmented by 1, he forgets), that is, 65536 bezants. Then the first square of the third row contains 2 chests, he asserts; even according to his own preceding text it should obviously be 1 chest. Going on with doublings he asserts that the second contains 4 instead of 2 chests, etc., until the last square of the fourth row, supposed to contain 65536 chests, reinterpreted as a *house*. Further, 65536 houses make up a *city*. The last square of the last row is then supposed – the same error persisting – to contain 65536 cities.

Another pedagogical illustration of the immensity of the number follows: if each unit represents a grain of wheat, identified with the weight unit *grain*, how many standard ships can be filled? The outcome, 1525028445[84] ships plus a fraction, is said to be “like infinite, and uncountable”.

A comparison with Abū Kāmil’s paraphrase of al-Khwārizmī explains Fibonacci’s mistake. Abū Kāmil [ed. trans. Rashed 2012: 726*f*] relates that

> Muḥammad ibn Mūsā [al-Khwārizmī] – may God be satisfied with him – has made this easy and accessible by saying: you put down the first, two, he put down the first as two in order to liberate himself from adding one; if he multiplies it with itself, one has four, which is the second. And if one multiplies four by itself, one has sixteen, which is the fourth. [...] If you want to double and add the squares of the chess-board, multiply the eighth, which is two hundred fifty-six, by itself. What you obtain is the sixteenth. Multiply the sixteenth square by itself, what you obtain is the thirty-second square. [...].

84 All manuscripts including **L** actually write 1725028445 [Giusti 2020: 488, apparatus; Giusti 2017: 206, apparatus].

There are no chests here, but we find the idea of using 16 squares, that is, two rows, as a basis for simplified calculation. Fibonacci, when borrowing either from Abū Kāmil or some later writing depending on him (or possibly some other source depending directly on al-Khwārizmī),[85] has obviously not only overlooked that his source starts with 2 in the first case but also not discovered that the consequences he draws from it are wrong and contradict what he has said just before. The mistakes are also in **L**, showing that Fibonacci did not make a complete critical reading of his master-copy when preparing the 1228 edition.

The "ship" illustration makes use of Pisa metrology. If Fibonacci did not devise it himself (as he may well have done) he will at least have had to recalculate.

Is the first part of the "other explanation" then Fibonacci's own?

Almost certainly not. For one thing, its basic trick is also in Abū Kāmil's text. Moreover, it was discussed in much more detail as a "practical way other than what most people are accustomed to do" by al-Uqlīdisī in Damascus in 952 [ed., trans. Saidan 1978: 338]; in 1449, al-Qalaṣādī [ed. trans. Souissi 1988: 75*f*] also described it, with the further observation that

> the number placed in the 9th [square] is equal to the sum of the numbers of the first 8 squares plus 1. [...] taking the square of the number in the 9th one gets the one in the 17th; taking the square of the latter one gets the one in the 33rd; doing the same with the latter one gets the number of the 65th, that is, the sum of the first 64 numbers plus 1, which is the first term.

So, the approach was widespread among Arabic mathematicians, and too close in the details to make us believe that Fibonacci made an independent exposition.

The alternative interpretation of the doubling problem ("the following square is the double of all its antecedents") determines this sequence by stepwise calculation:

$$1 - 2 - 6 - 18 - 54 - 162 - 486 - 1458 - 4374$$

Fibonacci does not point out that the step factor from 2 onward is constantly 3, as follows from inspection, and as can also easily be argued. In symbols, if square n holds C_n,

$$C_n = 2\cdot(1+...+C_{n-1}) ,$$

[85] Whatever the source, it must have been written – the details that connect Fibonacci to Abū Kāmil are not of the kind that would survive oral transmission.

then

$$C_{n+1} = 2\cdot(1+...+C_n) = 2C_n+2\cdot(1+...+C_{n-1}) = 2C_n+C_n = 3C_n \ .$$

It would be more lengthy but not much more difficult to formulate this in words. Fibonacci makes no attempt to do so, however. Instead he observes that

$$(1+2+6)^2 = 81 = 1+2+6+18+54$$

while

$$(1+2+6+18+54)^2 = 6561 = 1+2+6+16+54+162+486+1458+4374 \ .$$

This rule is claimed with no hint of an argument to go on corresponding to squares no. 5, 9, 17, 33 and 65. It is obviously a parallel to what was used in the "first interpretation", and would be evident if we knew the *sums* (not just the contents of the single squares) to be in geometric progression. In symbols, and taking into account that $C_{n+1} = 2\cdot3^{n-1}$, it is easily established that they are: from

$$C_{n+1} = 2\cdot\sum_1^n C_i$$

($n\geq2$) follows

$$\sum_1^n C_i = 3^{n-1} \ .$$

Moreover, for n taking on the values 5, 9, 17, 33 and 65, n–1 equals successive powers of two. A skilled medieval arithmetician should be able to establish it using words, perhaps (as *Elements* VII–IX) using by letter-carrying line segments. However, Fibonacci seems not to posses the building blocks for the argument, and therefore offers none. (If he had known how to do it, he would probably have inserted an explanation in his own words, in the style "if you want to understand why ...".)

So he merely uses the rule to find

$$\sum_1^{65} C_n = 3{,}433{,}683{,}820{,}292{,}512{,}484{,}657{,}849{,}089{,}281 \ ,$$

from which he concludes that

$$\sum_1^{64} C_n = \tfrac{1}{3}\cdot\sum_1^{65} C_n = 1{,}144{,}561{,}273{,}430{,}837{,}494{,}885{,}949{,}696{,}427 \ ,$$

Even the latter result is correct – namely (since $C_{n+1} = 3C_n$ for n≥2) because

$$\sum_1^{65} C_n = 1+2+\sum_3^{65} C_n = 3+3\sum_2^{64} C_n = 3\sum_1^{64} C_n \ .$$

Fibonacci's argument for this, however is quite opaque (should his words be meant as one).

Given the absence of genuine arguments, even this second interpretation of the chess-board problem and the way to deal with it must have been borrowed – for once apparently with modest understanding. The similarity of the 5–9–17–33–65-argument with what we saw in the treatment of the "first interpretation" indicates that Fibonacci took both from the same source – which then cannot be Abū Kāmil's own text.

From where he borrowed is a guess, but since we know that Abū Kāmil's *Algebra* circulated in al-Andalus in the 12th century; since the secondary stratum of chapter 14 and the bulk of part 15.1 appear to have come from there, al-Andalus should be the best guess.

A further borrowing, this one almost certainly taken over from al-Andalus, regards the problem of the "unknown heritage", or at least its sophisticated versions. I have dealt with this extensively in [Høyrup 2008], with further discussion in [Høyrup 2021a: 34–42], In consequence I shall restrict myself to summarizing, referring for documentation to these publications.

The basic variants of the problem type – actually its earliest known occurrence – runs like this [B279;G446]:

> Somebody coming to his end instructed the oldest of his sons, saying: Divide my possessions among yourself in this manner. You take one bezant, and the seventh of what is left; but to the next one of the sons he said, and you take 2 bezants, and the seventh of what is left. But to the next one, that he should take 3 bezants, and take control of $^1/_7$ of what was left. And in this way he called all his sons in order, giving each one more than the others; and afterwards always $^1/_7$ of what was left; the last however had the rest. It turned out, however, that all had equally of the possessions of the father on the said condition. It is asked, how many were the sons; and how much he owned. Indeed you do like this: for the seventh, which he gave to each, you retain 7; from which you extract 1, 6 remain. And so many were the sons; which 6 you multiplied in itself; and so many were his bezants. And if the first of the sons had had $^1/_7$ of the possessions of the father, and afterwards 1 bezant; and the second had had $^1/_7$ of the rest, and two bezants; and in this way it would have gone on for the other sons, adding for each one in order 1 bezant; then the sons would similarly be 6, and the bezants 6 seven times, that is 42. And if in each question the first should have 3 bezants, the second 6, and the rest similarly their bezants in ternary ascension; then the sons would similarly be 6, and the amount of the bezants would be the triple of the said amounts, that is, of 36 and of 42.

Since the problem appears not to have been known in the Arabic work at large and to be of late ancient Greek or Byzantine origin, the bezants of the question may reflect a borrowing from Byzantium. As we see, no argument is give for the validity of the solution.

A number of variants follow, which would lead to fractional sons, and where Fibonacci avoids the absurdity by dividing *a number* in corresponding ways – first the simple extrapolation where the fraction is $^{1}/_{11}$, which when understood as $^{1}/_{5\frac{1}{2}}$ allows use of the same rule. Then, however, [B279;G447] onward, come the really sophisticated variants where the fraction is $^{6}/_{31}$ and the absolute contributions 2, 5, 8, ... (taken either before and after the fraction) and $^{5}/_{19}$, with absolute contributions 3, 5, 7, ... For the first of these Fibonacci finds a solution by means of *regula recta,* positing the number to be a *thing* and using the equality of the two first parts. Logically, this merely shows that he finds the *only possible* solution; maybe for that reason Fibonacci presents a numerical proof. Next he claims to derive a general rule, which however does not follow without use of symbolic algebraic manipulations from Fibonacci's solution. Worse: while Fibonacci's *solution* translated into a rule or formula would also be valid for absolute contributions 3, 5, 7, ..., the *rule* he gives then involves negative numbers; the rule therefore has to be changed. Evidently Fibonacci has borrowed the rules for the four sophisticated cases from somewhere, being given no clue as to their foundation. He therefore solves one of the problems on his own, and claims that the rules he has borrowed come from there.

It appears that the only place from where Fibonacci can have acquired knowledge about the rules for the sophisticated version is al-Andalus or at least the Ibero-Provençal area; and since we have absolutely no traces of any kind of "mathematician" working in the Christian parts of this area before Fibonacci and possessing the competence required to derive such rules, once more al-Andalus seems to be the source for the inspiration.

The simple versions are in **L** [ed. Giusti 2017: 157], the others not; they have thus been added in 1228. It is therefore quite plausible that Fibonacci encountered the former orally in Byzantium, and took the latter from writing produced in al-Andalus (the rules given for the sophisticated versions are far too complex to have been transmitted orally).

In general it should be observed that everything mentioned above which Fibonacci states to have encountered in Byzantium or which appears to have had a Byzantine origin seems to belong to the 1202 version. In quite a few cases, as we have seen, Fibonacci elaborated with supplementary explanations in 1228.

In contrast it is worth observing that the 1202 prologue mentions nothing that could refer to the Iberian Peninsula including al-Andalus. That might seem strange, until we notice that all apparently Iberian loans we have discussed seem to have been borrowed from writings, not from face-to-face interaction; most of it moreover for the 1228 version – the "Castilian master" writing on barter being the most plausible exception.[86]

Vernacular versus magisterial

In the prologue Fibonacci promises to teach what he has learned on his visits to trading places,

> adding a few things from my own mind, and also putting in some subtleties of Euclid's art of geometry, I made an effort to compose, in as intelligible a fashion as I could.

Yet his ambitions transcend the mere presentation of practical arithmetic with some interspersed Euclid and a few of his own inventions. That becomes obvious if we look at a number of key terms – *vulgaris* ("vernacular" or "common"), *magistraliter* ("scholarly", "of school", and such), *nos* ("we", mostly absorbed in the verbal conjugation), with some variations of the expressions.

The opening passage of Fibonacci's *Pratica geometrie* [ed. Boncompagni 1862: 1] confronts *secundum demonstrationes geometricas*, "according to geometrical demonstrations", and *secundum vulgarem consuetudinem, quasi laicali more*, "according to the vernacular way, approximately the way of laymen". Variations on the same contraposition turn up time and again in the *Liber abbaci*.

Let us first remember note 33, about the introduction of the "rule of a tree" [B173;G296]. A tree is told to have $^1/_4\,^1/_3$ under ground, namely 21 palms.

> Because $^1/_4\,^1/_3$ are found in 12, understand the same tree to be divided into 12 parts, of which the third and fourth, that is, 7 parts, are 21 palms. Therefore, as 7 are to 21, so are proportionally 12 parts to the length of the tree. And because, when four numbers are proportional, the

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[86] The influence from Gerard on the beginning of chapter 15 part 1 may also be an exception; this influence, however, is exactly an *influence*, no faithful borrowing.

Regarding the rest of chapter 15 part 3, the absence of chronological cues makes it impossible to decide whether, for instance, the *avere* group was adopted in 1202 or in 1228.

multiplication of the first in the fourth is equal to the multiplication of the second in the third; therefore, if you multiply the second, 21, by the third, 12, which are known, and divide by the similarly known first number, that is, by 7, 36 results for the fourth, unknown number, that is, for the length of that tree.

After this perfectly argued solution by means of proportion theory follows

> there is indeed a different way which we use, that is, that you posit for the unknown thing some freely chosen known number which can be divided in integers by the fractions that are posited in the same question [...]. For example, the number asked for of this question is the length of the tree. Therefore posit it to be 12, which is divided integrally by 3 and by 4, which are below the strokes. And because it is said that $^{1}/_{4}\,^{1}/_{3}$ of the tree are 21, take $^{1}/_{4}\,^{1}/_{3}$ of the 12 that were posited: they will be 7, and if by accident they had been 21 we would certainly have what was proposed, namely that that tree would be 21 palms. But because 7 are not 21, they fall proportionally: as 7 is to 21, thus the posited tree is to the one asked for, namely 12 to 36. Therefore it is habitual to say, "for 12 which I posit come 7; what should I posit so that 21 come". And when it is said thus, the extreme numbers should be multiplied together, that is, 12 by 21, and the outcome should be divided by the remaining number.

That is, the initial scholarly formulation replaces what is habitual in Fibonacci's reference environment. And this habitual way is a single false position leading to application of the Rule of Three.

The corresponding problem in **L** [ed. Giusti 2017: 31] is slightly different – it states that 20 palms are above ground. That is a trivial difference. Non-trivial is the difference in approach: the scholarly method is absent, and therefore the second is just presented as it is – it goes by itself that this is the usual way. Fibonacci's "reference environment" is thus the one he encountered around the Mediterranean before 1202 (as we might expect).

What *we* do also turns up the beginning of chapter 11, dealing with alloying [B143;G249]. Here it is explained that

> *cum dicimus: habeo monetam ad uncias quantaslibet, ut dicamus ad 2, intelligimus quod in libra ipsius monete habeantur uncie 2 argenti*
>
> when we say, 'I have silver at whatever ounces, let us say at two, then we understand that in the pound of that same coin there are 2 ounces of silver.

This way of speaking is widespread in practical arithmetics from the Christian mediterranean. It is known from Byzantium, from the Iberian Peninsula, and not least from later Italian abbacus books – see [Høyrup

2019: 302]. However, it is *not* Fibonacci's own primary way to express himself. He does it when he wants to signal to his reader that a problem not dealing with alloying has *the mathematical structure* of an alloying problem and should be solved as one; and when, in the solution of alloying problems involving more than two types of silver, sub-problems are solved by means of the simple alloying model – that is, again, for signalling. The reader is thus expected to recognize the phrase and know what it refers to. Once again, it belongs to Fibonacci's reference environment.

The near-equivalence of "we" and "vernacular" is shown in the beginning of chapter 7 [B63;G107]. Here, the addition of $^1/_3$ and $^1/_4$ is explained first *secundum vulgum modum*, "in the vernacular way", which turns out to be very close to what above was presented as the way "we" deal with the tree problem: At first we find a number of which $^1/_3$ and $^1/_4$ can be taken (namely 3·4 = 12); the sum of these fractions of 12 are 7, whence the sum is (4+3)÷12. The alternative, not named but actually what was taught in schools and thus *magistraliter*, is

$$\frac{1}{4}+\frac{1}{3} = \frac{3\cdot 1+4\cdot 1}{3\cdot 4}\ ,$$

shown also in a diagram.

An explanation in chapter 8 of partnership calculation [B114*f*;G197*f*][87] is given *secundum pisanam consuetudinem*, "according to Pisa habit"; *secundum vulgarem modum*, "in the common way", and "secundum artem", "according to art". A capital of 152 £ earns a profit of 56 £, of which one fourth has to be detracted (for taxes or port fee, one must understand); 42 £ remain as net profit. The gain per invested £ is asked for.

The "Pisa way" makes the calculation (1·42)÷152, expression in the idiom of ascending continued fractions, where $^1/_{152}$ = $\frac{1\ \ 0}{8\ 19}$.[88] By an arithmetical trick that does not concern us here this is transformed into $\frac{6\ \ 6\ \ 5}{19\ 12\ 20}$, "that is ß 5 and δ 6 and almost a third of a δ".

In the "common way", 41 £ is first divided by 8, with outcome £ 5 and ß 5, which is then divided by 19; the result is stated with greater precision, as ß 5 and δ $6^6/_{19}$. Now the question is added, how much of the gain falls to an investment of 13 £, and it is found by multiplication by 13 in "the common way" – namely separate multiplication of the three addends 5 ß,

87 Anticipated, otherwise it is the topic of chapter 10.

88 That is, starting from the right in the Arabic way, $\frac{0}{19}+\frac{1}{8\cdot 19}$. The principle can be extended *ad libitum* toward the left (Fibonacci elsewhere goes until 10 levels).

6 δ and $^{6}/_{19}$.

The calculation “according to art” is similar to the Pisa way, and can be summarized as (13·42)÷152, followed by the arithmetical trick that allows the expression of the outcome in terms of ß and δ. It corresponds to the normal partnership rule, which can be seen as application of the Rule of Three in parallel. This rule is the topic of the following chapter 8; Fibonacci never gives it a name except [B83;G141] *maior guisa*, “the major mode” to find the price of goods involving four proportional numbers. At the present location in chapter 7 he does not even mention proportionality. It appears that the calculation according to art was the way that was taught in schools that justified the Rule of Three by means of proportion theory (at the time not yet Italian abbacus schools[89]), whereas the “common way” was that of merchant reckoners who preferred to keep it simple and in as far as possible to avoid too difficult mixed numbers.

The absence of proportion language here suggests that the present confrontation of “common” and “by art” goes back to 1202. A parallel [B204;G342] which does not go back to 1202 is found in a second alternative by *regula recta* to the problem which served above to show that Fibonacci knew the *regula recta* already in 1202 (above, p. 16). This second solution by means of that rule, absent from **L** where it should have been [ed. Giusti 2017: 78*f*], involves a multiplication of $21^{1}/_{3}{}^{1}/_{10}{}^{1}/_{24}{}^{1}/_{32}$ by 20. Fibonacci explains that *vulgariter* it is made via multiplication 20·21, $20\cdot{}^{1}/_{3}$, $20\cdot{}^{1}/_{10}$, etc., without indicating or naming the alternative.[90]

There are more instances, also with explicit confrontation. A final example to discuss is the determination of $4+\sqrt{\sqrt{10}}$ [B364;G563] in chapter 14 part 3, already spoken of above, p. 28. The *a*-*b*-*c* line diagram used for the magisterial way emulates the very beginning of part 3 [B361;G559], which shows on the basis of the same line diagram (equally lettered *a*-*b*-*c*) that the square on a binomial is itself a binomial, promising to show in

[89] We have no direct evidence for how it was taught to merchant youth in Byzantium and the Arabic commercial cities, but in both cases a basis in elementary proportion principles seems likely.

[90] The magisterial way would obviously be to transform $21^{1}/_{3}{}^{1}/_{10}{}^{1}/_{24}{}^{1}/_{32}$ into $^{10423}/_{480}$. When it comes to actual computation Fibonacci appears to have understood the practical advantage of the vernacular way – but he still feels obliged to name it.

the following that it is a first binomial. As we remember from p. 28, this was first shown in part 2 (new in the 1228 version) and then a second time in part 3, indicating that it was already in the early version of the chapter. We may reasonably presume that the magisterial determination of 4+√√10 was also there.

All in all it turns out that the pair *vulgariter / magistraliter* was present in the *Liber abbaci* as devised in 1202. Extensive new magisterial matters were certainly added in 1228 – probably, as argued, parts 2a and 2b of chapter 14, and part 1 of chapter 15, but there the characterizing term is not mentioned (there are indeed no "vernacular" counterparts).

Though promising at first merely to teach that he had learned, adding something from Euclid and something of his own, Fibonacci's original project went further, transforming what he had learned into "real mathematics" – as said further on in the prologue, "demonstrating almost everything that I have included by a firm proof". That is what he did at least in principle. For instance, as we have seen, when presenting the rule of the single false position in connection with the tree problem Fibonacci gives a magisterial rationalization; afterwards, once this is done, he allows himself many false-position solutions (giving proofs only when homogeneous higher-degree problems are solved by means of a false position). Sometimes, as we have seen, the proofs are of Fibonacci's own making; more often they appear to have been borrowed, but not always together with the matter to be proved.

All in all, Fibonacci's ambition was thus, as expressed by Felix Klein [1908], to teach *Elementarmathematik von höheren Standpunkte aus*, "elementary [and not so elementary, but that holds for Klein too] mathematics from a higher vantage point".

A fragmentary summary

It is impossible to sum up all the observations made above – that is prevented by the nature of the material, where most of the text is "just mathematics" and not transparent as to where it comes from and how it came about. None the less, a number of main results can be listed.

First of all, Fibonacci conserved a master copy of the *Liber abbaci* from the beginning which was the foundation for later revisions (perhaps one, perhaps two or more but then not very different) commonly considered as from 1228. He never made a thorough revision of the text but merely added and deleted where he found it adequate.[91]

According to the prologue (not contradicted by the observations here made), much, probably most of the work is copied from varied sources. Fibonacci took care to copy faithfully, even when it comes to the terminology of his sources. As the Latin 12th-century mathematical translators he also conserved the lettering of diagrams. He mostly took care to make his explanatory commentaries separately, avoiding pastiche. This was no absolute principle of his, however – at times the lettering of diagrams show that he has intervened in a borrowed proof.

We have no reason to doubt the claim in the prologue that much of the material at least in chapters 1 through 13 was adopted from what Fibonacci had encountered during the commercial travels of his youth, and thus probably in oral interaction. Other ingredients come from written sources, often (we do not know *how* often) taken from pre-existing Latin translations that have now disappeared – many of them with varying degrees of certainty prepared on the basis of originals created in al-Andalus (during the 12th-century Indian summer of al-Andalus scholarship, since they seem not to have influenced Arabic mathematics at large). The geographical origin of such written sources (or their very existence) is never betrayed by Fibonacci, except his reference to the "Castilian master", apparently made in the 1202 version and later eliminated.

In chapter 14, borrowings from written sources made for the 1228 version ultimately going back to *Elements* X have given rise to a secondary stratum. In chapter 15, part 1 appears to have been similarly inserted as a whole on that occasion. To which extent something similar happened to the questions section of chapter 15 part 3 is unclear, but it seems a fair guess that at least the "extended *avere* group" a late-comer.

In chapter 13, it is similarly a reasonable guess that the proof of the second method of double false was inserted in 1228. However, since the argument builds on the use of line diagrams it can be no more – line diagrams are certainly used profusely in the material inserted in 1228, but at least in chapter 14 part 3 line diagrams appear to have been used in

[91] Very strictly speaking this only follows from the time the prototype for **L** chapter 12 was prepared. *If* a first revision was made for Frederick II, and *if* **L** as well as other extant manuscripts all descend from that revision, we cannot get behind it. But if so, this first revision will hardly have been thorough, and **L** chapter 12 and its archetype will still be close to the 1202 text. I shall allow myself to speak as if the shared archetype for all manuscripts including **L** chapter 12 was the 1202 text.

material going back to 1202.

From the beginning, Fibonacci's aim was to present the material *magistraliter,* at least in the sense that he showed how "vernacular" way could be rationalized according to scholarly norms though then he would follow these ways rather freely – more or less as a quantum physicist who, once it is explained how Dirac's δ-function, while paradoxical if understood as a function, can be rationalized as distribution, goes on using it freely.

Was the Fibonacci we know from the *Liber abbaci* a "great mathematician"? That depends on definitions. Was Euclid a "great mathematician"? Even he borrowed much; according to widespread assumptions *Elements* V came from Eudoxos and *Elements* X from Theaitetos, and to which extent Euclid transformed the material we cannot know. At least, Euclid was an enormously influential mathematician.

Fibonacci had no similar influence – not because he was not sufficiently great but because he was *too* great for his environment. Apart from Frederick's court philosophers he had no audience capable to learn from the advanced material he presented in his own times (and the main interest of these philosophers lay elsewhere). Fibonacci got competent readers in the mid- to late 15th century, not least Benedetto da Firenze and Luca Pacioli; but by then it was too late, mathematics was going other ways. that, however, is a different story, and I shall leave documentation to other occasions.

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